# ON THE EQUIVALENCE OF PERMUTATION POLYNOMIAL INTERLEAVERS OF DEGREE HIGHER THAN TWO AND ARP INTERLEAVERS FOR TURBO CODES

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**Abstract -** Three of the most common and performant interleavers for Turbo Codes (TCs) are Dithered Relative Prime (DRP) interleavers, Quadratic Permutation Polynomial (QPP) interleavers and Almost Regular Permutation (ARP) interleavers. Recently, in [1] it was shown that DRP and QPP interleavers can be expressed in the ARP interleaver function form. In this paper, the conditions for a QPP interleaver to be equivalent to an ARP interleaver are extended for an interleaver based on Permutation Polynomials (PP) of any degree. PP interleavers of degree 3, 4, and 5, whose coefficients respect the sufficient conditions from [2], are always equivalent to an ARP interleaver with disorder degree smaller than the interleaver length (when it is not a prime number).

*Keywords: turbo codes, ARP interleaver, PP interleaver, equivalence.*

# 1. Introduction

Two of the most known and performant interleavers for Turbo Codes (TCs) are the Almost Regular Permutation (ARP) interleavers [3] and the permutation polynomial (PP) interleavers, introduced by Sun and Takeshita in [4-5].

Recently, in [1], it was shown that Dithered Relative Prime (DRP) [6] and Quadratic Permutation Polynomial (QPP) interleavers can be expressed in the ARP interleaver function form. As a consequence, this proves that ARP interleaves can obtain at least the same minimum distance as DRP and QPP interleavers. These equivalences present the advantage of a unified implementation, when these different interleavers are used for some specific applications.

QPP interleavers have been intensively studied in [4-5, 7-18]. However, in [9, 11, 19-20] it was shown that PP interleavers of degree greater than 2 may lead to performances superior to those of QPP. This motivates analyzing the equivalence of PP and ARP interleavers.

Necessary and sufficient conditions for generating QPPs and Cubic Permutation Polynomials (CPPs) were given in [5, 21-22], but not for PPs of degree higher than 3. The sufficient conditions that must be satisfied by the coefficients of a polynomial to be PP were given in [2]. In this paper, we present the conditions to express interleavers based on any degree PPs as ARP interleavers. Customizations are made for PPs of degrees 3, 4 or 5, whose coefficients comply the sufficient conditions in [2]. We prove that these PPs can always be expressed as ARP interleavers, having a disorder degree lower than interleaver's length, when it is not a prime number. Specific examples are shown in each case.

The paper is structured as follows. In Section 2 the mathematical model for ARP and PP interleavers of degree *m* is presented. Section 3 provides the sufficient conditions for a *m*-PP interleaver to be expressed as an ARP interleaver. These conditions are written for *m*=2, *m*=3, *m*=4 and *m*=5 in subsections 3.1, 3.2, 3.3 and 3.4, respectively. In Section 4 the sufficient conditions from Section 3, subsections 3.2, 3.3, 3.4, are applied for PP interleavers of degree 3, 4 and 5, whose coefficients comply the sufficient conditions in [2], in subsections 4.2, 4.3 and 4.4, respectively. Finally, Section 5 presents some conclusions and remarks regarding Parallel Linear Permutation Polynomial (PLPP) based interleavers composed of $L$ LPPs introduced in [23].

# 2. Mathematical model for ARP and PP interleavers of degree *m*

## 2.1. ARP Interleavers

The ARP interleaver was proposed by Berrou et al. in [3]. Its interleaving function is defined as:

$$\Pi_{ARP}(i) = \left(P \cdot i + S_{(i \bmod Q)}\right) \bmod K, \qquad (1)$$

where $i = 0, \ldots, K-1$ denotes the address of the data symbol after interleaving and $\Pi_{ARP}(i)$ represents its corresponding address before interleaving. $P$ is a positive integer relatively prime to the interleaver size, $K$. The disorder cycle or disorder degree in the permutation is denoted by $Q$, which corresponds to the number of shifts in $S$. $K$ must be a multiple of $Q$.

### 2.2. PP interleavers of degree *m*

PP interleavers were proposed by Sun and Takeshita [4-5]. They are based on permutation polynomials over the integer ring $\mathbb{Z}_K$, where $K$ is the interleaver length. For a PP of degree $m$, denoted *m*-PP, the permutation function is defined as

$$\Pi_{m-PP}(i) = \left( f_1 \cdot i + f_2 \cdot i^2 + f_3 \cdot i^3 + \ldots + f_m \cdot i^m \right) \bmod K, \qquad (2)$$

where $i = 0, \ldots, K-1$ denotes the address of a data symbol after interleaving and $\Pi_{m-PP}(i)$ represents its address before interleaving. A sufficient condition for generating PPs of degree $m$ is given in Theorem 1 from [2].

## 3. Sufficient conditions for a *m*-PP interleaver to be expressed as an ARP interleaver

As in [1], a sufficient condition for the existence of an ARP-equivalent form of a valid *m*-PP interleaver is that the following equations hold:

$$\begin{cases} (P \cdot i) \bmod K = (f_1 \cdot i) \bmod K & (3) \\ S_{(i \bmod Q)} \bmod K = \left( f_2 \cdot i^2 + f_3 \cdot i^3 + \ldots + f_m \cdot i^m \right) \bmod K & (4) \end{cases}$$

The above equations are satisfied if:

$$P = f_1, \qquad (5)$$

$$\left( f_2 \cdot i^2 + f_3 \cdot i^3 + \ldots + f_m \cdot i^m \right) \bmod K =$$
$$= \left( f_2 \cdot (i+Q)^2 + f_3 \cdot (i+Q)^3 + \ldots + f_m \cdot (i+Q)^m \right) \bmod K \qquad (6)$$

Considering the formula of the binomial theorem,

$$(i+Q)^n = \sum_{k=0}^{n} C_n^k \cdot i^k \cdot Q^{n-k} = \sum_{k=0}^{n-1} C_n^k \cdot i^k \cdot Q^{n-k} + i^n, \qquad (7)$$

(6) is true for:

$$f_2 \cdot \sum_{k=0}^{1} C_2^k \cdot i^k \cdot Q^{2-k} + f_3 \cdot \sum_{k=0}^{2} C_3^k \cdot i^k \cdot Q^{3-k} + \ldots + f_m \cdot \sum_{k=0}^{m-1} C_m^k \cdot i^k \cdot Q^{m-k} = 0 \bmod K$$

(8)

or

$$i^0 \cdot \left( f_2 \cdot C_2^0 \cdot Q^2 + f_3 \cdot C_3^0 \cdot Q^3 + \ldots + f_m \cdot C_m^0 \cdot Q^m \right) +$$
$$+ i^1 \cdot \left( f_2 \cdot C_2^1 \cdot Q^1 + f_3 \cdot C_3^1 \cdot Q^2 + \ldots + f_m \cdot C_m^1 \cdot Q^{m-1} \right) +$$
$$+ i^2 \cdot \left( f_3 \cdot C_3^2 \cdot Q^1 + f_4 \cdot C_4^2 \cdot Q^2 + \ldots + f_m \cdot C_m^2 \cdot Q^{m-2} \right) +$$
$$+ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots +$$
$$+ i^{m-2} \cdot \left( f_{m-1} \cdot C_{m-1}^{m-2} \cdot Q^1 + f_m \cdot C_m^{m-2} \cdot Q^2 \right) + i^{m-1} \cdot f_m \cdot C_m^{m-1} \cdot Q^1 = 0 \bmod K \qquad (9)$$

or

$$\begin{cases}\left(f_2\cdot Q^2+f_3\cdot Q^3+\ldots+f_m\cdot Q^m\right)\bmod K=0\\ \left(2f_2\cdot Q+3f_3\cdot Q^2+\ldots+m\cdot f_m\cdot Q^{m-1}\right)\bmod K=0\\ \left(C_3^2\cdot f_3\cdot Q+C_4^2\cdot f_4\cdot Q^2+\ldots+C_m^2\cdot f_m\cdot Q^{m-1}\right)\bmod K=0\\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\ \left(C_{m-1}^{m-2}\cdot f_{m-1}\cdot Q+C_m^{m-2}\cdot f_m\cdot Q^2\right)\bmod K=0\\ \left(C_m^{m-1}\cdot f_m\cdot Q\right)\bmod K=0\end{cases}\tag{10}$$

The last equation from (10) implies that:

$$m\cdot f_m\cdot Q=l\cdot K,\, l\in\mathbb{N}^+,\tag{11}$$

which is satisfied for:

$$Q=\frac{l\cdot K}{m\cdot f_m},\, l\in\mathbb{N}^+.\tag{12}$$

Equations 2, 3, ..., *m*-1 from (10) can be compactly written as:

$$\left(\sum_{k'=k}^{m}C_{k'}^{k-1}\cdot f_{k'}\cdot Q^{k'-(k-1)}\right)\bmod K=0\text{, where } k=2,\ldots,m-1\tag{13}$$

Substituting (12) into (13), we have:

$$\left(\sum_{k'=k}^{m}C_{k'}^{k-1}\cdot f_{k'}\cdot\left(\frac{l\cdot K}{m\cdot f_m}\right)^{k'-(k-1)}\right)\bmod K=0,\tag{14}$$

which is true if

$$\sum_{k'=k}^{m}C_{k'}^{k-1}\cdot f_{k'}\cdot\left(\frac{l}{m\cdot f_m}\right)^{k'-(k-1)}\cdot K^{k'-k}\in\mathbb{N}^+\tag{15}$$

Considering (12), the first equation in (10) is true if

$$\sum_{k'=2}^{m}f_{k'}\cdot\left(\frac{l}{m\cdot f_m}\right)^{k'}\cdot K^{k'-1}\in\mathbb{N}^+.\tag{16}$$

As we will focus on PPs of degree 3, 4 or 5, we will write in detail the equation from (15) for $k=m-1$, $k=m-2$, $k=m-3$.

For $k=m-1$, (15) is equivalent to:

$$\frac{(m-1)\cdot l\cdot(2f_{m-1}+l\cdot K)}{2\cdot m\cdot f_m}\in\mathbb{N}^+\tag{17}$$

For $k=m-2$, (15) is equivalent to:

$$\frac{m\cdot(m-2)\cdot f_{m-2}\cdot f_m\cdot l+\frac{m\cdot(m-1)\cdot(m-2)}{2}\cdot f_{m-1}\cdot f_m\cdot l^2\cdot K}{(m\cdot f_m)^2}+$$

$$+\frac{\frac{(m-1)\cdot(m-2)}{6}\cdot l^3\cdot K^2}{(m\cdot f_m)^2}\in\mathbb{N}^+\tag{18}$$

For $k=m-3$, (15) is equivalent to:

$$\frac{m^2\cdot(m-3)\cdot f_{m-3}\cdot f_m^2\cdot l+\frac{(m-2)\cdot(m-3)}{2}\cdot m\cdot f_{m-2}\cdot f_m\cdot l^2\cdot K}{(m\cdot f_m)^3}+$$

$$+\frac{\frac{(m-1)\cdot(m-2)\cdot(m-3)}{6}\cdot f_{m-1}\cdot l^3\cdot K^2+\frac{(m-1)\cdot(m-2)\cdot(m-3)}{24}\cdot l^4\cdot K}{(m\cdot f_m)^2}\in\mathbb{N}^+$$

(19)

Further on, we customize the conditions (12), (16) and (17)-(19) for $m$=2, $m$=3, $m$=4 or $m$=5.

### 3.1. Sufficient conditions for a QPP interleaver to be expressed as an ARP interleaver

In this case $m$=2 (i.e. the PP is a QPP) and the system (10) has only two equations, (12) and (16).

(12) becomes

$Q=\frac{l\cdot K}{2\cdot f_2}$, $l\in\mathbb{N}^+$, which is the same to (25) in [1],

and (16) becomes

$f_2\cdot\left(\frac{l}{2\cdot f_2}\right)^2\cdot K\in\mathbb{N}^+$, which is the same as the condition after eq. (32) in [1].

### 3.2. Sufficient conditions for a CPP interleaver to be expressed as an ARP interleaver

In this case $m$=3 (i.e. the PP is a CPP) and the system (10) has three equations, (12), (16) and (17).

(12) becomes

$$Q=\frac{l\cdot K}{3\cdot f_3},\ l\in\mathbb{N}^+, \qquad (20)$$

(16) becomes

$$\frac{l^2\cdot K\cdot(3\cdot f_2+l\cdot K)}{3^3\cdot f_3^2}\in\mathbb{N}^+, \qquad (21)$$

and (17) becomes

$$\frac{l\cdot(2\cdot f_2+l\cdot K)}{3\cdot f_3}\in\mathbb{N}^+. \qquad (22)$$

### 3.3. Sufficient conditions for a 4-PP interleaver to be expressed as an ARP interleaver

In this case $m$=4 and the system (10) has four equations, (12), (16), (17) and (18).

(12) becomes

$$Q=\frac{l\cdot K}{4\cdot f_4},\ l\in\mathbb{N}^+ \qquad (23)$$

(16) becomes

$$\frac{l^2\cdot K\cdot(2^4\cdot f_2\cdot f_4+2^2\cdot f_3\cdot l\cdot K+l^2\cdot K^2)}{2^8\cdot f_4^3}\in\mathbb{N}^+, \qquad (24)$$

(17) becomes

$$\frac{3\cdot l\cdot\left(2\cdot f_3+l\cdot K\right)}{2^3\cdot f_4}\in\mathbb{N}^+. \qquad (25)$$

and (18) becomes

$$\frac{l\cdot\left(2^3\cdot f_2\cdot f_4+2^2\cdot 3\cdot f_3\cdot f_4\cdot l\cdot K+l^2\cdot K^2\right)}{2^4\cdot f_4^2}\in\mathbb{N}^+ \qquad (26)$$

### 3.4. Sufficient conditions for a 5-PP interleaver to be expressed as an ARP interleaver

In this case $m$=5 and the system (10) has five equations, (12), (16), (17), (18) and (19).

(12) becomes

$$Q=\frac{l\cdot K}{5\cdot f_5},\ l\in\mathbb{N}^+, \qquad (27)$$

(16) becomes

$$\frac{l^2\cdot K\cdot\left(5^3\cdot f_2\cdot f_5^2+5^2\cdot f_3\cdot f_5\cdot l\cdot K+5\cdot f_4\cdot l^2\cdot K^2+l^3\cdot K^3\right)}{5^5\cdot f_5^4}\in\mathbb{N}^+, \qquad (28)$$

(17) becomes

$$\frac{2\cdot l\cdot\left(2\cdot f_4+l\cdot K\right)}{5\cdot f_5}\in\mathbb{N}^+, \qquad (29)$$

(18) becomes

$$\frac{l\cdot\left(5\cdot 3\cdot f_3\cdot f_5+5\cdot 2\cdot 3\cdot f_4\cdot f_5\cdot l\cdot K+2\cdot l^2\cdot K^2\right)}{5^2\cdot f_5^2}\in\mathbb{N}^+, \qquad (30)$$

and (19) becomes

$$\frac{l\cdot\left(5^2\cdot 2\cdot f_2\cdot f_5^2+5\cdot 3\cdot f_3\cdot f_5\cdot l\cdot K+4\cdot f_4\cdot l^2\cdot K^2+l^3\cdot K^3\right)}{5^3\cdot f_5^3}\in\mathbb{N}^+. \qquad (31)$$

One of the sufficient conditions imposed in [2] on a polynomial's coefficients, so that it is PP irrespective of the value of $K$, is for $f_2, f_3,\ldots, f_m$ to be multiples of each odd prime number that divides $K$.

We consider the decomposition of $K$ in prime factors [1] as:

$$K=2^{\alpha_{K,1}}\cdot\prod_{i=2}^{\omega(K)}p_i^{\alpha_{K,i}}, \qquad (32)$$

the decomposition of the coefficients $f_j,\ j=2,\ldots,m$, as

$$f_j=2^{\alpha_{f_j,1}}\cdot\prod_{i=2}^{\omega(K)}p_i^{\alpha_{f_j,i}}\cdot\prod_{i=\omega(K)+1}^{\omega(f_j)}p_{i,f_j}^{\alpha_{f_j,i}} \qquad (33)$$

and the decomposition of $l$ from (12) as

$$l=2^{\alpha_{l,1}}\cdot\prod_{i=2}^{\omega(K)}p_i^{\alpha_{l,i}}\cdot\prod_{i=\omega(K)+1}^{\omega(f_m)}p_{i,f_m}^{\alpha_{l,i}}, \qquad (34)$$

where $\omega(K)$ and $\omega(f_j)$ represent the number of different prime factors from the decomposition of $K$ and $f_j,\ j=2,\ldots,m$, respectively.

# 4. PP interleavers of degree 3, 4 or 5 seen as peculiar cases of ARP interleavers

Let us rewrite conditions (20) - (31), taking into account (32) - (34). In order to be consistent with the numbering of subsections for PP interleavers of degree 2, 3, 4 and 5, respectively, the following 4 subsections correspond to the four ones in Section 3.

## 4.1. QPP interleavers seen as peculiar cases of ARP interleavers

The case of QPP interleavers was studied in [1] and we skip it.

## 4.2. CPP interleavers seen as peculiar cases of ARP interleavers

From (20), we have

$$Q = \frac{l \cdot K}{3 \cdot f_3} = 2^{\alpha_{l,1}+\alpha_{K,1}-\alpha_{f_3,1}} \cdot 3^{-1} \cdot \prod_{i=2}^{\omega(K)} p_i^{\alpha_{l,i}+\alpha_{K,i}-\alpha_{f_3,i}} \cdot \prod_{i=\omega(K)+1}^{\omega(f_3)} p_{i,f_3}^{\alpha_{l,i}-\alpha_{f_3,i}} \quad (35)$$

In this case we have to consider the factorization of $K$, of the coefficients $f_2$ and $f_3$, and of $l$, depending on how $3 \nmid K$ or $3 | K$.

Thus, if $3 \nmid K$, $K$ and $f_2$ are as in (32) and (33), respectively, and

$$f_3 = 2^{\alpha_{f_3,1}} \cdot \prod_{i=2}^{\omega(K)} p_i^{\alpha_{f_3,i}} \cdot 3^{\alpha_{f_3,\omega(K)+1}} \cdot \prod_{i=\omega(K)+2}^{\omega(f_3)} p_{i,f_3}^{\alpha_{f_3,i}} \quad (36)$$

$$l = 2^{\alpha_{l,1}} \cdot \prod_{i=2}^{\omega(K)} p_i^{\alpha_{l,i}} \cdot 3^{\alpha_{l,\omega(K)+1}} \cdot \prod_{i=\omega(K)+2}^{\omega(f_3)} p_{i,f_3}^{\alpha_{l,i}} \quad (37)$$

If $3 | K$, $f_2$ is as in (33), and

$$K = 2^{\alpha_{K,1}} \cdot 3^{\alpha_{K,2}} \cdot \prod_{i=3}^{\omega(K)} p_i^{\alpha_{K,i}} \quad (38)$$

$$f_3 = 2^{\alpha_{f_3,1}} \cdot 3^{\alpha_{f_3,2}} \cdot \prod_{i=3}^{\omega(K)} p_i^{\alpha_{f_3,i}} \cdot \prod_{i=\omega(K)+1}^{\omega(f_3)} p_{i,f_3}^{\alpha_{f_3,i}} \quad (39)$$

$$l = 2^{\alpha_{l,1}} \cdot 3^{\alpha_{l,2}} \cdot \prod_{i=3}^{\omega(K)} p_i^{\alpha_{l,i}} \cdot \prod_{i=\omega(K)+1}^{\omega(f_3)} p_{i,f_3}^{\alpha_{l,i}} \quad (40)$$

We mention that in (32)-(33), (36) and (38)-(39) we could have $\alpha_{K,1} = 0$ and/or $\alpha_{f_2,1} = 0$ and/or $\alpha_{f_3,1} = 0$ and/or $\alpha_{f_3,\omega(K)+1} = 0$, the rest of prime numbers' powers being grater than 0.

Then, if $3 \nmid K$, we have

$$Q = 2^{\alpha_{l,1}+\alpha_{K,1}-\alpha_{f_3,1}} \cdot \prod_{i=2}^{\omega(K)} p_i^{\alpha_{l,i}+\alpha_{K,i}-\alpha_{f_3,i}} \cdot 3^{\alpha_{l,\omega(K)+1}-\alpha_{f_3,\omega(K)+1}-1} \cdot \prod_{i=\omega(K)+2}^{\omega(f_3)} p_{i,f_3}^{\alpha_{l,i}-\alpha_{f_3,i}} \quad (41)$$

and if $3 | K$

$$Q = 2^{\alpha_{l,1}+\alpha_{K,1}-\alpha_{f_3,1}} \cdot 3^{\alpha_{l,2}+\alpha_{K,2}-\alpha_{f_3,2}-1} \cdot \prod_{i=3}^{\omega(K)} p_i^{\alpha_{l,i}+\alpha_{K,i}-\alpha_{f_3,i}} \cdot \prod_{i=\omega(K)+1}^{\omega(f_3)} p_{i,f_3}^{\alpha_{l,i}-\alpha_{f_3,i}} \quad (42)$$

Since $Q$ has to be a divisor of $K$, from (35) we have

I) If $3 \nmid K$

$$0 \le \alpha_{l,i} + \alpha_{K,i} - \alpha_{f_3,i} \le \alpha_{K,i} \Leftrightarrow \alpha_{f_3,i} - \alpha_{K,i} \le \alpha_{l,i} \le \alpha_{f_3,i} \qquad (43)$$

for $i = 1,2,\ldots,\omega(K)$,

$$\alpha_{l,\omega(K)+1} = \alpha_{f_3,\omega(K)+1} + 1 \qquad (44)$$

and

$$\alpha_{l,i} = \alpha_{f_3,i}, \qquad (45)$$

for $i = \omega(K)+2,\ldots,\omega(f_3)$.

II) If $3 \mid K$

$$\alpha_{f_3,i} - \alpha_{K,i} \le \alpha_{l,i} \le \alpha_{f_3,i} \qquad (46)$$

for $i = 1,3,4,\ldots,\omega(K)$,

$$\alpha_{f_3,2} - \alpha_{K,2} + 1 \le \alpha_{l,2} \le \alpha_{f_3,2} + 1 \qquad (47)$$

and

$$\alpha_{l,i} = \alpha_{f_3,i}, \qquad (48)$$

for $i = \omega(K)+1,\ldots,\omega(f_3)$.

From (21) we have

I) If $3 \nmid K$, considering (44)-(45), we have:

$$\frac{l^2 \cdot K \cdot (3 \cdot f_2 + l \cdot K)}{3^3 \cdot f_3^2} = 2^{2\alpha_{l,1}+\alpha_{K,1}+\alpha_{f_2,1}-2\alpha_{f_3,1}} \cdot 3^{-2} \cdot \prod_{i=2}^{\omega(K)} p_i^{2\alpha_{l,i}+\alpha_{K,i}+\alpha_{f_2,i}-2\alpha_{f_3,i}} \cdot 3^{2\alpha_{l,\omega(K)+1}-2\alpha_{f_3,\omega(K)+1}} \cdot$$

$$\cdot \prod_{i=\omega(K)+1}^{\omega(f_2)} p_{i,f_2}^{\alpha_{f_2,i}} + 2^{3\alpha_{l,1}+2\alpha_{K,1}-2\alpha_{f_3,1}} \cdot 3^{-3} \cdot \prod_{i=2}^{\omega(K)} p_i^{3\alpha_{l,i}+2\alpha_{K,i}-2\alpha_{f_3,i}} \cdot 3^{3\alpha_{l,\omega(K)+1}-2\alpha_{f_3,\omega(K)+1}} \cdot \prod_{i=\omega(K)+2}^{\omega(f_3)} p_{i,f_3}^{\alpha_{f_3,i}} =$$

$$= 2^{2\alpha_{l,1}+\alpha_{K,1}-2\alpha_{f_3,1}} \cdot \prod_{i=2}^{\omega(K)} p_i^{2\alpha_{l,i}+\alpha_{K,i}-2\alpha_{f_3,i}} \cdot \left\{ 2^{\alpha_{f_2,1}} \cdot \prod_{i=2}^{\omega(K)} p_i^{\alpha_{f_2,i}} \cdot \prod_{i=\omega(K)+1}^{\omega(f_2)} p_{i,f_2}^{\alpha_{f_2,i}} + \right.$$

$$\left. +2^{\alpha_{l,1}+\alpha_{K,1}} \cdot \prod_{i=2}^{\omega(K)} p_i^{\alpha_{l,i}+\alpha_{K,i}} \cdot 3^{\alpha_{f_3,\omega(K)+1}} \cdot \prod_{i=\omega(K)+2}^{\omega(f_3)} p_{i,f_3}^{\alpha_{f_3,i}} \right\} \in \mathbb{N}^+$$

(49)

A sufficient condition for (49) to be a positive natural number is:

$$2\alpha_{l,i} + \alpha_{K,i} - 2\alpha_{f_3,i} + \min\left\{\alpha_{f_2,i}, \alpha_{l,i} + \alpha_{K,i}\right\} \ge 0$$

$$\Leftrightarrow \alpha_{l,i} \ge \alpha_{f_3,i} - \frac{\alpha_{K,i} + \min\left\{\alpha_{f_2,i}, \alpha_{l,i} + \alpha_{K,i}\right\}}{2}, \qquad (50)$$

for $i = 1,2,\ldots,\omega(K)$.

II) If $3 \mid K$, considering (48), we have:

$$\frac{l^2 \cdot K \cdot (3 \cdot f_2 + l \cdot K)}{3^3 \cdot f_3^2} = 2^{2\alpha_{l,1}+\alpha_{K,1}+\alpha_{f_2,1}-2\alpha_{f_3,1}} \cdot 3^{2\alpha_{l,2}+\alpha_{K,2}+\alpha_{f_4,2}-2\alpha_{f_3,2}} \cdot 3^{-2} \cdot \prod_{i=3}^{\omega(K)} p_i^{2\alpha_{l,i}+\alpha_{K,i}+\alpha_{f_2,i}-2\alpha_{f_3,i}} \cdot$$

$$\cdot \prod_{i=\omega(K)+1}^{\omega(f_2)} p_{i,f_2}^{\alpha_{f_2,i}} + 2^{3\alpha_{l,1}+2\alpha_{K,1}-2\alpha_{f_3,1}} \cdot 3^{3\alpha_{l,2}+2\alpha_{K,2}-2\alpha_{f_3,2}} \cdot 3^{-3} \cdot \prod_{i=3}^{\omega(K)} p_i^{3\alpha_{l,i}+2\alpha_{K,i}-2\alpha_{f_3,i}} \cdot \prod_{i=\omega(K)+1}^{\omega(f_3)} p_{i,f_3}^{\alpha_{f_3,i}} =$$

$$= 2^{2\alpha_{l,1}+\alpha_{K,1}-2\alpha_{f_3,1}} \cdot 3^{2\alpha_{l,2}+\alpha_{K,2}-2\alpha_{f_3,2}-2} \cdot \prod_{i=3}^{\omega(K)} p_i^{2\alpha_{l,i}+\alpha_{K,i}-2\alpha_{f_3,i}} \cdot \left\{ 2^{\alpha_{f_2,1}} \cdot 3^{\alpha_{f_4,2}} \cdot \prod_{i=3}^{\omega(K)} p_i^{\alpha_{f_2,i}} \cdot \prod_{i=\omega(K)+1}^{\omega(f_2)} p_{i,f_2}^{\alpha_{f_2,i}} + \right.$$

$$\left. + 2^{\alpha_{l,1}+\alpha_{K,1}} \cdot 3^{\alpha_{l,2}+\alpha_{K,2}-1} \cdot \prod_{i=3}^{\omega(K)} p_i^{\alpha_{l,i}+\alpha_{K,i}} \cdot \prod_{i=\omega(K)+1}^{\omega(f_3)} p_{i,f_3}^{\alpha_{f_3,i}} \right\} \in \mathbb{N}^+$$

(51)

A sufficient condition for (51) to be a positive natural number is:

$$2\alpha_{l,i} + \alpha_{K,i} - 2\alpha_{f_3,i} + \min\{\alpha_{f_2,i}, \alpha_{l,i} + \alpha_{K,i}\} \geq 0$$

$$\Leftrightarrow \alpha_{l,i} \geq \alpha_{f_3,i} - \frac{\alpha_{K,i} + \min\{\alpha_{f_2,i}, \alpha_{l,i} + \alpha_{K,i}\}}{2}, \qquad (52)$$

for $i = 1, 3, 4, \ldots, \omega(K)$ and

$$2\alpha_{l,2} + \alpha_{K,2} - 2\alpha_{f_3,2} - 2 + \min\{\alpha_{f_2,2}, \alpha_{l,2} + \alpha_{K,2} - 1\} \geq 0$$

$$\Leftrightarrow \alpha_{l,2} \geq \alpha_{f_3,2} + 1 - \frac{\alpha_{K,2} + \min\{\alpha_{f_2,2}, \alpha_{l,2} + \alpha_{K,2} - 1\}}{2}. \qquad (53)$$

From (22) we have

I) If $3 \nmid K$, considering (44)-(45), we have:

$$\frac{l \cdot (2 \cdot f_2 + l \cdot K)}{3 \cdot f_3} = 2^{\alpha_{l,1}+\alpha_{f_2,1}-\alpha_{f_3,1}+1} \cdot 3^{-1} \cdot \prod_{i=2}^{\omega(K)} p_i^{\alpha_{l,i}+\alpha_{f_2,i}-\alpha_{f_3,i}} \cdot 3^{\alpha_{l,\omega(K)+1}-\alpha_{f_3,\omega(K)+1}} \cdot \prod_{i=\omega(K)+1}^{\omega(f_2)} p_{i,f_2}^{\alpha_{f_2,i}} +$$

$$+ 2^{2\alpha_{l,1}+\alpha_{K,1}-\alpha_{f_3,1}} \cdot 3^{-1} \cdot \prod_{i=2}^{\omega(K)} p_i^{2\alpha_{l,i}+\alpha_{K,i}-\alpha_{f_3,i}} \cdot 3^{2\alpha_{l,\omega(K)+1}-\alpha_{f_3,\omega(K)+1}} \cdot \prod_{i=\omega(K)+2}^{\omega(f_3)} p_{i,f_3}^{\alpha_{f_3,i}} =$$

$$= 2^{\alpha_{l,1}-\alpha_{f_3,1}} \cdot \prod_{i=2}^{\omega(K)} p_i^{\alpha_{l,i}-\alpha_{f_3,i}} \cdot \left\{ 2^{\alpha_{f_2,1}+1} \cdot \prod_{i=2}^{\omega(K)} p_i^{\alpha_{f_2,i}} \cdot \prod_{i=\omega(K)+1}^{\omega(f_2)} p_{i,f_2}^{\alpha_{f_2,i}} + \right.$$

$$\left. + 2^{\alpha_{l,1}+\alpha_{K,1}} \cdot \prod_{i=2}^{\omega(K)} p_i^{\alpha_{l,i}+\alpha_{K,i}} \cdot 3^{\alpha_{f_3,\omega(K)+1}+1} \cdot \prod_{i=\omega(K)+2}^{\omega(f_3)} p_{i,f_3}^{\alpha_{f_3,i}} \right\} \in \mathbb{N}^+$$

(54)

A sufficient condition for (54) to be a positive natural number is:

$$\alpha_{l,1} - \alpha_{f_3,1} + \min\{\alpha_{f_2,1} + 1, \alpha_{l,1} + \alpha_{K,1}\} \geq 0 \quad \Leftrightarrow \alpha_{l,1} \geq \alpha_{f_3,1} - \min\{\alpha_{f_2,1} + 1, \alpha_{l,1} + \alpha_{K,1}\}$$

(55)

and

$$\alpha_{l,i} - \alpha_{f_3,i} + \min\{\alpha_{f_2,i}, \alpha_{l,i} + \alpha_{K,i}\} \geq 0 \;\Leftrightarrow \alpha_{l,i} \geq \alpha_{f_3,i} - \min\{\alpha_{f_2,i}, \alpha_{l,i} + \alpha_{K,i}\} \qquad (56)$$

for $i = 2, 3, \ldots, \omega(K)$.

II) If $3 \mid K$, considering (48), we have:

$$\frac{l\cdot(2\cdot f_2+l\cdot K)}{3\cdot f_3}=2^{\alpha_{l,1}+\alpha_{f_2,1}-\alpha_{f_3,1}+1}\cdot 3^{\alpha_{l,2}+\alpha_{f_2,2}-\alpha_{f_3,2}}\cdot 3^{-1}\cdot\prod_{i=3}^{\omega(K)}p_i^{\alpha_{l,i}+\alpha_{f_2,i}-\alpha_{f_3,i}}\cdot\prod_{i=\omega(K)+1}^{\omega(f_2)}p_{i,f_2}^{\alpha_{f_2,i}}+$$

$$+2^{2\alpha_{l,1}+\alpha_{K,1}-\alpha_{f_3,1}}\cdot 3^{2\alpha_{l,2}+\alpha_{K,2}-\alpha_{f_3,2}}\cdot 3^{-1}\cdot\prod_{i=3}^{\omega(K)}p_i^{2\alpha_{l,i}+\alpha_{K,i}-\alpha_{f_3,i}}\cdot\prod_{i=\omega(K)+1}^{\omega(f_3)}p_{i,f_3}^{\alpha_{f_3,i}}=$$

$$=2^{\alpha_{l,1}-\alpha_{f_3,1}}\cdot 3^{\alpha_{l,2}-\alpha_{f_3,2}-1}\cdot\prod_{i=3}^{\omega(K)}p_i^{\alpha_{l,i}-\alpha_{f_3,i}}\cdot\left\{2^{\alpha_{f_2,1}+1}\cdot 3^{\alpha_{f_2,2}}\cdot\prod_{i=3}^{\omega(K)}p_i^{\alpha_{f_2,i}}\cdot\prod_{i=\omega(K)+1}^{\omega(f_2)}p_{i,f_2}^{\alpha_{f_2,i}}+\right.$$

$$\left.+2^{\alpha_{l,1}+\alpha_{K,1}}\cdot 3^{\alpha_{l,2}+\alpha_{K,2}}\cdot\prod_{i=3}^{\omega(K)}p_i^{\alpha_{l,i}+\alpha_{K,i}}\cdot\prod_{i=\omega(K)+1}^{\omega(f_3)}p_{i,f_3}^{\alpha_{f_3,i}}\right\}\in\mathbb{N}^+$$

(57)

A sufficient condition for (57) to be a positive natural number is:

$$\alpha_{l,1}-\alpha_{f_3,1}+\min\left\{\alpha_{f_2,1}+1,\alpha_{l,1}+\alpha_{K,1}\right\}\geq 0 \quad\Leftrightarrow \alpha_{l,1}\geq\alpha_{f_3,1}-\min\left\{\alpha_{f_2,1}+1,\alpha_{l,1}+\alpha_{K,1}\right\}$$

(58)

$$\alpha_{l,2}-\alpha_{f_3,2}-1+\min\left\{\alpha_{f_2,2},\alpha_{l,2}+\alpha_{K,2}\right\}\geq 0\Leftrightarrow \alpha_{l,2}\geq\alpha_{f_3,2}+1-\min\left\{\alpha_{f_2,2},\alpha_{l,2}+\alpha_{K,2}\right\}$$

(59)

$$\alpha_{l,i}-\alpha_{f_3,i}+\min\left\{\alpha_{f_2,i},\alpha_{l,i}+\alpha_{K,i}\right\}\geq 0 \;\Leftrightarrow \alpha_{l,i}\geq\alpha_{f_3,i}-\min\left\{\alpha_{f_2,i},\alpha_{l,i}+\alpha_{K,i}\right\} \qquad (60)$$

for $i=3,4,\ldots,\omega(K)$.

In conclusion, if $3\nmid K$, rejoining the conditions (43)-(45), (50), and (55)-(56), we have:

$$\alpha_{f_3,1}-\min\left\{\alpha_{K,1},\min\left\{\alpha_{f_2,1}+1,\alpha_{l,1}+\alpha_{K,1}\right\},\frac{\alpha_{K,1}+\min\left\{\alpha_{f_2,1},\alpha_{l,1}+\alpha_{K,1}\right\}}{2}\right\}\leq\alpha_{l,1}\leq\alpha_{f_3,1}$$

(61)

$$\alpha_{f_3,i}-\min\left\{\alpha_{K,i},\min\left\{\alpha_{f_2,i},\alpha_{l,i}+\alpha_{K,i}\right\},\frac{\alpha_{K,i}+\min\left\{\alpha_{f_2,i},\alpha_{l,i}+\alpha_{K,i}\right\}}{2}\right\}\leq\alpha_{l,i}\leq\alpha_{f_3,i}$$

(62)

for $i=2,3,\ldots,\omega(K)$,

$$\alpha_{l,\omega(K)+1}=\alpha_{f_3,\omega(K)+1}+1 \qquad (63)$$

and

$$\alpha_{l,i}=\alpha_{f_3,i}, \qquad (64)$$

for $i=\omega(K)+2,\ldots,\omega(f_3)$.

If $3\mid K$, rejoining the conditions (46)-(48), (52)-(53), and (58)-(60), we have:

$$\alpha_{f_3,1}-\min\left\{\alpha_{K,1},\min\left\{\alpha_{f_2,1}+1,\alpha_{l,1}+\alpha_{K,1}\right\},\frac{\alpha_{K,1}+\min\left\{\alpha_{f_2,1},\alpha_{l,1}+\alpha_{K,1}\right\}}{2}\right\}\leq\alpha_{l,1}\leq\alpha_{f_3,1}$$

(65)

$$\alpha_{f_3,2}+1-\min\left\{\alpha_{K,2},\frac{\alpha_{K,2}+\min\left\{\alpha_{f_2,2},\alpha_{l,2}+\alpha_{K,2}-1\right\}}{2},\min\left\{\alpha_{f_2,2},\alpha_{l,2}+\alpha_{K,2}\right\}\right\}\leq$$

$$\leq\alpha_{l,2}\leq\alpha_{f_3,2}+1 \qquad (66)$$

$$\alpha_{f_3,i} - \min\left\{\alpha_{K,i}, \min\left\{\alpha_{f_2,i}, \alpha_{l,i} + \alpha_{K,i}\right\}, \frac{\alpha_{K,i} + \min\left\{\alpha_{f_2,i}, \alpha_{l,i} + \alpha_{K,i}\right\}}{2}\right\} \le \alpha_{l,i} \le \alpha_{f_3,i}$$

(67)

for $i = 3, 4, \ldots, \omega(K)$

and

$$\alpha_{l,i} = \alpha_{f_3,i}, \qquad (68)$$

for $i = \omega(K)+1, \ldots, \omega(f_3)$.

We mention that if the left hand side of the double inequalities (61)-(62) and (65)-(67), is negative, it is considered to be equal to 0. Since the minimum value of quantities in the brackets from previous relations are nonnegative, for any CPP satisfying the conditions of theorem 1 from [2] there is an equivalent ARP.

Example 1:

Let $K = 1696 = 2^5 \cdot 53^1$ and the CPP coefficients $f_1 = 55 = 5^1 \cdot 11^1$, $f_2 = 954 = 2^1 \cdot 53^1 \cdot 3^2$ and $f_3 = 1272 = 2^3 \cdot 53^1 \cdot 3^1$. These coefficients respect conditions a), c) and d) from theorem 1 in [2], resulting in a valid CPP.

The values of $l$ leading to valid values for $Q$ are factorized as follows:

$$l = 2^{\alpha_{l,1}} \cdot 53^{\alpha_{l,2}} \cdot 3^{\alpha_{l,3}}, \qquad (69)$$

Because $3 \nmid K$, we impose the conditions (61)-(63).

From (63), we have

$$\alpha_{l,3} = 2, \qquad (70)$$

so that the factorization of $Q$ is

$$Q = 2^{\alpha_{l,1}+2} \cdot 53^{\alpha_{l,2}}. \qquad (71)$$

Condition (61) can be written as:

$$3 - \min\left\{5, \min\left\{1+1, \alpha_{l,1}+5\right\}, \frac{5 + \min\left\{1, \alpha_{l,1}+5\right\}}{2}\right\} \le \alpha_{l,1} \le 3$$

$$\Leftrightarrow 3 - \min\left\{5, 2, \frac{5+1}{2}\right\} \le \alpha_{l,1} \le 3 \Leftrightarrow 3 - 2 \le \alpha_{l,1} \le 3 \Leftrightarrow 1 \le \alpha_{l,1} \le 3. \qquad (72)$$

Condition (62) can be written as:

$$1 - \min\left\{1, \min\left\{1, \alpha_{l,2}+1\right\}, \frac{1 + \min\left\{1, \alpha_{l,2}+1\right\}}{2}\right\} \le \alpha_{l,2} \le 1 \Leftrightarrow 1 - 1 \le \alpha_{l,2} \le 1$$

$$\Leftrightarrow 0 \le \alpha_{l,2} \le 1. \qquad (73)$$

From (72) and (73), we see that there are 6 possible values for $l$ and, consequently, for $Q$, given in Table 1. The possible values for the vector $S$ for this interleaver, for which $Q_s = 8$ (the minimum value of $Q$) are given in Table 2.

**Table 1.** Possible values for $Q$ in the CPP with $K = 1696$, $f_1 = 55$, $f_2 = 954$ and $f_3 = 1272$.

| $l$ | 18 | 36 | 72 | 954 | 1908 | 3816 |
|---|---|---|---|---|---|---|
| $Q$ | 8 | 16 | 32 | 424 | 848 | 1696 |

**Table 2**. Equivalent ARP interleavers with $P = f_1$, $Q_s = 8$, $S(0) = S(4) = 0$.

| $K$ | $f_1$ | $f_2$ | $f_3$ | $S(1)$ | $S(2)$ | $S(3)$ | $S(5)$ | $S(6)$ | $S(7)$ |
|---|---|---|---|---|---|---|---|---|---|
| 1696 | 55 | 954 | 1272 | 530 | 424 | 530 | 1378 | 424 | 1378 |

Example 2:

Let $K = 216 = 2^3 \cdot 3^3$ and the CPP coefficients $f_1 = 11$, $f_2 = 36 = 2^2 \cdot 3^2$ and $f_3 = 144 = 2^4 \cdot 3^2$. These coefficients respect conditions a), c) and d) from theorem 1 in [2], resulting in a valid CPP.

The values of $l$ leading to valid values for $Q$ are factorized as follows:

$$l = 2^{\alpha_{l,1}} \cdot 3^{\alpha_{l,2}}, \qquad (74)$$

so that the factorization of $Q$ is

$$Q = 2^{\alpha_{l,1}-1} \cdot 3^{\alpha_{l,2}}. \qquad (75)$$

Because $3 \mid K$, we impose the conditions (65)-(66).

Condition (65) can be written as:

$$4 - \min\left\{3, \min\{2+1, \alpha_{l,1}+3\}, \frac{3+\min\{2, \alpha_{l,1}+3\}}{2}\right\} \le \alpha_{l,1} \le 4$$

$$\Leftrightarrow 4 - \min\left\{3, 3, \frac{3+2}{2}\right\} \le \alpha_{l,1} \le 4 \Leftrightarrow 4 - 2.5 \le \alpha_{l,1} \le 4 \Leftrightarrow 2 \le \alpha_{l,1} \le 4 \qquad (76)$$

From (66) we have:

$$2+1-\min\left\{3, \frac{3+\min\{2, \alpha_{l,2}+3-1\}}{2}, \min\{2, \alpha_{l,2}+3\}\right\} \le \alpha_{l,2} \le 2+1$$

$$\Leftrightarrow 3 - \min\left\{3, \frac{5}{2}, 2\right\} \le \alpha_{l,2} \le 3 \Leftrightarrow 3-2 \le \alpha_{l,2} \le 3 \Leftrightarrow 1 \le \alpha_{l,2} \le 3. \qquad (77)$$

From (76) - (77), we see that there are 9 possible values for $l$ and, consequently, for $Q$, given in Table 3. The possible values for the vector $S$ for this interleaver, for which $Q_s = 6$ (the minimum value of $Q$) are given in Table 4.

**Table 3**. Possible values for $Q$ in the CPP with $K = 216$, $f_1 = 11$, $f_2 = 36$ and $f_3 = 144$.

| $l$ | 12 | 24 | 36 | 48 | 72 | 108 | 144 | 216 | 432 |
|---|---|---|---|---|---|---|---|---|---|
| $Q$ | 6 | 12 | 18 | 24 | 36 | 54 | 72 | 108 | 216 |

**Table 4**. Equivalent ARP interleavers with $P = f_1$, $Q_s = 6$, $S(0) = S(2) = 0$.

| $K$ | $f_1$ | $f_2$ | $f_3$ | $S(1)$ | $S(3)$ | $S(4)$ | $S(5)$ |
|---|---|---|---|---|---|---|---|
| 216 | 11 | 36 | 144 | 180 | 108 | 72 | 108 |

Example 3:

Let $K = 4802 = 2^1 \cdot 7^4$ and the CPP coefficients $f_1 = 10 = 2^1 \cdot 5^1$, $f_2 = 70 = 2^1 \cdot 5^1 \cdot 7^1$ and $f_3 = 7 = 7^1$. These coefficients respect conditions b) and c) from theorem 1 in [2], resulting in a valid CPP.

The values of $l$ leading to valid values for $Q$ are factorized as follows:

$$l = 2^{\alpha_{l,1}} \cdot 7^{\alpha_{l,2}} \cdot 3^{\alpha_{l,3}}, \qquad (78)$$

In the following relations, we will consider $\alpha_{f_2,3} = 0$ and $\alpha_{f_3,1} = \alpha_{f_3,3} = 0$.

Because $3 \nmid K$, we impose the conditions (61)-(63).

From condition (63), we have

$$\alpha_{l,3} = 0 + 1 = 1, \qquad (79)$$

so that the factorization of $Q$ is

$$Q = 2^{\alpha_{l,1}+1} \cdot 7^{\alpha_{l,2}+3}. \qquad (80)$$

Condition (61) can be written as

$$0 - \min\left\{1, \min\left\{1+1, \alpha_{l,1}+1\right\}, \frac{1+\min\left\{1, \alpha_{l,1}+1\right\}}{2}\right\} \le \alpha_{l,1} \le 0$$

$$\Leftrightarrow 0 - 1 \le \alpha_{l,1} \le 0 \Leftrightarrow \alpha_{l,1} = 0. \qquad (81)$$

Condition (62) can be written as

$$1 - \min\left\{4, \min\left\{1, \alpha_{l,2}+4\right\}, \frac{4+\min\left\{1, \alpha_{l,2}+4\right\}}{2}\right\} \le \alpha_{l,2} \le 1 \Leftrightarrow 1 - 1 \le \alpha_{l,2} \le 1$$

$$\Leftrightarrow 0 \le \alpha_{l,2} \le 1. \qquad (82)$$

From (81)-(82), we see that there are only 2 possible values for $l$ and, consequently, for $Q$, given in Table 5. As the minimum value of $Q$ is too large, namely $Q_s = 686$, we have not given the possible values for the vector $S$ for this interleaver.

**Table 5**. Possible values for $Q$ in the CPP with $K = 4802$, $f_1 = 10$, $f_2 = 70$ and $f_3 = 7$.

| $l$ | 3 | 21 |
|---|---|---|
| $Q$ | 686 | 4802 |

Example 4:

Let $K = 1225 = 5^2 \cdot 7^2$ and the CPP coefficients $f_1 = 12 = 2^2 \cdot 3^1$, $f_2 = 525 = 3^1 \cdot 5^2 \cdot 7^1$ and $f_3 = 140 = 2^2 \cdot 5^1 \cdot 7^1$. These coefficients respect conditions a) and c) from theorem 1 in [2], resulting in a valid CPP.

The values of $l$ leading to valid values for $Q$ are factorized as follows:

$$l = 2^{\alpha_{l,1}} \cdot 5^{\alpha_{l,2}} \cdot 7^{\alpha_{l,3}} \cdot 3^{\alpha_{l,4}}, \qquad (83)$$

In the following relations, we will consider $\alpha_{K,1} = 0$, $\alpha_{f_2,1} = 0$ and $\alpha_{f_3,4} = 0$.

Because $3 \nmid K$, we impose the conditions (61)-(63).

From (63) we have

$$\alpha_{l,4} = 0 + 1 = 1, \qquad (84)$$

so that the factorization of $Q$ is

$$Q = 2^{\alpha_{l,1}-2} \cdot 5^{\alpha_{l,2}+1} \cdot 7^{\alpha_{l,3}+1}. \qquad (85)$$

Condition (61) can be written as:

$$2 - \min\left\{0, \min\left\{0+1, \alpha_{l,1}+0\right\}, \frac{0+\min\left\{0, \alpha_{l,1}+0\right\}}{2}\right\} \le \alpha_{l,1} \le 2$$

$\Leftrightarrow 2-0 \le \alpha_{l,1} \le 2 \Leftrightarrow \alpha_{l,1} = 2$. (86)

Condition (62) can be written as:

$$1-\min\left\{2, \min\{2, \alpha_{l,2}+2\}, \frac{2+\min\{2, \alpha_{l,2}+2\}}{2}\right\} \le \alpha_{l,2} \le 1 \Leftrightarrow 1-2 \le \alpha_{l,2} \le 1$$

$\Leftrightarrow 0 \le \alpha_{l,2} \le 1$ (87)

and

$$1-\min\left\{2, \min\{1, \alpha_{l,3}+2\}, \frac{2+\min\{1, \alpha_{l,3}+2\}}{2}\right\} \le \alpha_{l,3} \le 1 \Leftrightarrow 1-1 \le \alpha_{l,3} \le 1$$

$\Leftrightarrow 0 \le \alpha_{l,3} \le 1$, (88)

respectively.

From (86)-(88), we see that there are 4 possible values for $l$ and, consequently, for $Q$, given in Table 6. As the minimum value of $Q$ is too large, namely $Q_s = 35$, we have not given the possible values for the vector $S$ for this interleaver.

**Table 6**. Possible values for $Q$ in the CPP with $K = 1225$, $f_1 = 12$, $f_2 = 525$ and $f_3 = 140$.

| $l$ | 14700 | 73500 | 102900 | 514500 |
|---|---|---|---|---|
| $Q$ | 35 | 175 | 245 | 1225 |

### 4.3. 4-PP interleavers seen as peculiar cases of ARP interleavers

From (20), we have

$$Q = \frac{l \cdot K}{4 \cdot f_4} = 2^{\alpha_{l,1}+\alpha_{K,1}-\alpha_{f_4,1}-2} \cdot \prod_{i=2}^{\omega(K)} p_i^{\alpha_{l,i}+\alpha_{K,i}-\alpha_{f_4,i}} \cdot \prod_{i=\omega(K)+1}^{\omega(f_4)} p_{i,f_4}^{\alpha_{l,i}-\alpha_{f_4,i}} \quad (89)$$

We mention that in relations (32)-(33) we could have $\alpha_{K,1} = 0$ and/or $\alpha_{f_2,1} = 0$ and/or $\alpha_{f_3,1} = 0$ and/or $\alpha_{f_4,1} = 0$, the rest of the powers of the prime numbers being greater than 0.

As $Q$ has to be a divisor of *K*, from (89), we have:

$$0 \le \alpha_{l,1} + \alpha_{K,1} - \alpha_{f_4,1} - 2 \le \alpha_{K,1} \Leftrightarrow \alpha_{f_4,1} - \alpha_{K,1} + 2 \le \alpha_{l,1} \le \alpha_{f_4,1} + 2 \quad (90)$$

$$0 \le \alpha_{l,i} + \alpha_{K,i} - \alpha_{f_4,i} \le \alpha_{K,i} \Leftrightarrow \alpha_{f_4,i} - \alpha_{K,i} \le \alpha_{l,i} \le \alpha_{f_4,i} \quad (91)$$

for $i = 2, 3, \ldots, \omega(K)$ and

$$\alpha_{l,i} = \alpha_{f_4,i}, \quad (92)$$

for $i = \omega(K)+1, \ldots, \omega(f_4)$.

From (24), considering (92), we have

$$\frac{l^2 \cdot K \cdot \left(2^4 \cdot f_2 \cdot f_4 + 2^2 \cdot f_3 \cdot l \cdot K + l^2 \cdot K^2\right)}{2^8 \cdot f_4^3} = 2^{2\alpha_{l,1}+\alpha_{K,1}+\alpha_{f_2,1}-2\alpha_{f_4,1}-4} \cdot \prod_{i=2}^{\omega(K)} p_i^{2\alpha_{l,i}+\alpha_{K,i}+\alpha_{f_2,i}-2\alpha_{f_4,i}} \cdot$$

$$\cdot \prod_{i=\omega(K)+1}^{\omega(f_2)} p_{i,f_2}^{\alpha_{f_2,i}} \cdot \prod_{i=\omega(K)+1}^{\omega(f_4)} p_{i,f_4}^{2\alpha_{l,i}-2\alpha_{f_4,i}} + 2^{3\alpha_{l,1}+2\alpha_{K,1}+\alpha_{f_3,1}-3\alpha_{f_4,1}-6} \cdot \prod_{i=2}^{\omega(K)} p_i^{3\alpha_{l,i}+2\alpha_{K,i}+\alpha_{f_3,i}-3\alpha_{f_4,i}} \cdot$$

$$\cdot \prod_{i=\omega(K)+1}^{\omega(f_3)} p_{i,f_3}^{\alpha_{f_3,i}} \cdot \prod_{i=\omega(K)+1}^{\omega(f_4)} p_{i,f_4}^{3\alpha_{l,i}-3\alpha_{f_4,i}} + 2^{4\alpha_{l,1}+3\alpha_{K,1}-3\alpha_{f_4,1}-8} \cdot \prod_{i=2}^{\omega(K)} p_i^{4\alpha_{l,i}+3\alpha_{K,i}-3\alpha_{f_4,i}} \cdot \prod_{i=\omega(K)+1}^{\omega(f_4)} p_{i,f_4}^{3\alpha_{l,i}-3\alpha_{f_4,i}} =$$

$$= 2^{2\alpha_{l,1}+\alpha_{K,1}-3\alpha_{f_4,1}-8} \cdot \prod_{i=2}^{\omega(K)} p_i^{2\alpha_{l,i}+\alpha_{K,i}-3\alpha_{f_4,i}} \cdot \left\{ 2^{\alpha_{f_2,1}+\alpha_{f_4,1}+4} \cdot \prod_{i=2}^{\omega(K)} p_i^{\alpha_{f_2,i}+\alpha_{f_4,i}} \cdot \prod_{i=\omega(K)+1}^{\omega(f_2)} p_{i,f_2}^{\alpha_{f_2,i}} + \right.$$

$$\left. + 2^{\alpha_{l,1}+\alpha_{K,1}+\alpha_{f_3,1}+2} \cdot \prod_{i=2}^{\omega(K)} p_i^{\alpha_{l,i}+\alpha_{K,i}+\alpha_{f_3,i}} \cdot \prod_{i=\omega(K)+1}^{\omega(f_3)} p_{i,f_3}^{\alpha_{f_3,i}} + 2^{2\alpha_{l,1}+2\alpha_{K,1}} \cdot \prod_{i=2}^{\omega(K)} p_i^{2\alpha_{l,i}+2\alpha_{K,i}} \right\} \in \mathbb{N}^+$$

(93)

A sufficient condition for (93) to be a positive natural number is

$$2\alpha_{l,1} + \alpha_{K,1} - 3\alpha_{f_4,1} - 8 +$$
$$+ \min\left\{\alpha_{f_2,1} + \alpha_{f_4,1} + 4, \alpha_{l,1} + \alpha_{K,1} + \alpha_{f_3,1} + 2, 2\alpha_{l,1} + 2\alpha_{K,1}\right\} \geq 0$$
$$\Leftrightarrow \alpha_{l,1} \geq \frac{3\alpha_{f_4,1}}{2} + 4 -$$
$$- \frac{\alpha_{K,1} + \min\left\{\alpha_{f_2,1} + \alpha_{f_4,1} + 4, \alpha_{l,1} + \alpha_{K,1} + \alpha_{f_3,1} + 2, 2\alpha_{l,1} + 2\alpha_{K,1}\right\}}{2} \qquad (94)$$

$$2\alpha_{l,i} + \alpha_{K,i} - 3\alpha_{f_4,i} + \min\left\{\alpha_{f_2,i} + \alpha_{f_4,i}, \alpha_{l,i} + \alpha_{K,i} + \alpha_{f_3,i}, 2\alpha_{l,i} + 2\alpha_{K,i}\right\} \geq 0$$
$$\Leftrightarrow \alpha_{l,i} \geq \frac{3\alpha_{f_4,i} - \alpha_{K,i} - \min\left\{\alpha_{f_2,i} + \alpha_{f_4,i}, \alpha_{l,i} + \alpha_{K,i} + \alpha_{f_3,i}, 2\alpha_{l,i} + 2\alpha_{K,i}\right\}}{2}, \quad (95)$$

for $i = 2,3,\ldots,\omega(K)$.

From (25), considering (92), we have:

$$\frac{3 \cdot l \cdot (2 \cdot f_3 + l \cdot K)}{2^3 \cdot f_4} = 2^{\alpha_{l,1}+\alpha_{f_3,1}-\alpha_{f_4,1}-2} \cdot 3 \cdot \prod_{i=2}^{\omega(K)} p_i^{\alpha_{l,i}+\alpha_{f_3,i}-\alpha_{f_4,i}} \cdot$$

$$\cdot \prod_{i=\omega(K)+1}^{\omega(f_3)} p_{i,f_3}^{\alpha_{f_3,i}} \cdot \prod_{i=\omega(K)+1}^{\omega(f_4)} p_{i,f_4}^{\alpha_{l,i}-\alpha_{f_4,i}} + 2^{2\alpha_{l,1}+\alpha_{K,1}-\alpha_{f_4,1}-3} \cdot 3 \cdot \prod_{i=2}^{\omega(K)} p_i^{2\alpha_{l,i}+\alpha_{K,i}-\alpha_{f_4,i}} \cdot \prod_{i=\omega(K)+1}^{\omega(f_4)} p_{i,f_4}^{2\alpha_{l,i}-\alpha_{f_4,i}} =$$

$$= 2^{\alpha_{l,1}-\alpha_{f_4,1}-3} \cdot 3 \cdot \prod_{i=2}^{\omega(K)} p_i^{\alpha_{l,i}-\alpha_{f_4,i}} \cdot$$

$$\left\{ 2^{\alpha_{f_3,1}+1} \cdot \prod_{i=2}^{\omega(K)} p_i^{\alpha_{f_3,i}} \cdot \prod_{i=\omega(K)+1}^{\omega(f_3)} p_{i,f_3}^{\alpha_{f_3,i}} + 2^{\alpha_{l,1}+\alpha_{K,1}} \cdot \prod_{i=2}^{\omega(K)} p_i^{\alpha_{l,i}+\alpha_{K,i}} \cdot \prod_{i=\omega(K)+1}^{\omega(f_4)} p_{i,f_4}^{\alpha_{f_4,i}} \right\} \in \mathbb{N}^+$$

(96)

A sufficient condition for (96) to be a positive natural number is:

$$\alpha_{l,1} - \alpha_{f_4,1} - 3 + \min\left\{\alpha_{f_3,1} + 1, \alpha_{l,1} + \alpha_{K,1}\right\} \geq 0$$
$$\Leftrightarrow \alpha_{l,1} \geq \alpha_{f_4,1} + 3 - \min\left\{\alpha_{f_3,1} + 1, \alpha_{l,1} + \alpha_{K,1}\right\} \quad (97)$$
$$\alpha_{l,i} - \alpha_{f_4,i} + \min\left\{\alpha_{f_3,i}, \alpha_{l,i} + \alpha_{K,i}\right\} \geq 0 \ \Leftrightarrow \alpha_{l,i} \geq \alpha_{f_4,i} - \min\left\{\alpha_{f_3,i}, \alpha_{l,i} + \alpha_{K,i}\right\}, \ (98a)$$

for $i = 2,3,\ldots,\omega(K)$, when $p_2 \neq 3$. If $p_2 = 3$, (98a) becomes

$$\alpha_{l,2} - \alpha_{f_4,2} + 1 + \min\left\{\alpha_{f_3,2}, \alpha_{l,2} + \alpha_{K,2}\right\} \geq 0$$

$$\Leftrightarrow \alpha_{l,2} \geq \alpha_{f_4,2} - 1 - \min\left\{\alpha_{f_3,2}, \alpha_{l,2} + \alpha_{K,2}\right\}. \quad (98b)$$

From (26), considering (92), we have

$$\frac{l\cdot\left(2^3\cdot f_2\cdot f_4 + 2^2\cdot 3\cdot f_3\cdot f_4\cdot l\cdot K + l^2\cdot K^2\right)}{2^4\cdot f_4^2} \in \mathbb{N}^+ = 2^{\alpha_{l,1}+\alpha_{f_2,1}-\alpha_{f_4,1}-1}\cdot\prod_{i=2}^{\omega(K)} p_i^{\alpha_{l,i}+\alpha_{f_2,i}-\alpha_{f_4,i}}\cdot$$

$$\cdot\prod_{i=\omega(K)+1}^{\omega(f_2)} p_{i,f_2}^{\alpha_{f_2,i}}\cdot\prod_{i=\omega(K)+1}^{\omega(f_4)} p_{i,f_4}^{\alpha_{l,i}-\alpha_{f_4,i}} + 2^{2\alpha_{l,1}+\alpha_{K,1}+\alpha_{f_3,1}-\alpha_{f_4,1}-2}\cdot 3\cdot\prod_{i=2}^{\omega(K)} p_i^{2\alpha_{l,i}+\alpha_{K,i}+\alpha_{f_3,i}-\alpha_{f_4,i}}\cdot$$

$$\cdot\prod_{i=\omega(K)+1}^{\omega(f_3)} p_{i,f_3}^{\alpha_{f_3,i}}\cdot\prod_{i=\omega(K)+1}^{\omega(f_4)} p_{i,f_4}^{2\alpha_{l,i}-\alpha_{f_4,i}} + 2^{3\alpha_{l,1}+2\alpha_{K,1}-2\alpha_{f_4,1}-4}\cdot\prod_{i=2}^{\omega(K)} p_i^{3\alpha_{l,i}+2\alpha_{K,i}-2\alpha_{f_4,i}}\cdot\prod_{i=\omega(K)+1}^{\omega(f_4)} p_{i,f_4}^{3\alpha_{l,i}-2\alpha_{f_4,i}} =$$

$$= 2^{\alpha_{l,1}-2\alpha_{f_4,1}-4}\cdot\prod_{i=2}^{\omega(K)} p_i^{\alpha_{l,i}-2\alpha_{f_4,i}}\cdot\left\{2^{\alpha_{f_2,1}+\alpha_{f_4,1}+3}\cdot\prod_{i=2}^{\omega(K)} p_i^{\alpha_{f_2,i}+\alpha_{f_4,i}}\cdot\prod_{i=\omega(K)+1}^{\omega(f_2)} p_{i,f_2}^{\alpha_{f_2,i}} +\right.$$

$$+2^{\alpha_{l,1}+\alpha_{K,1}+\alpha_{f_3,1}+\alpha_{f_4,1}+2}\cdot 3\cdot\prod_{i=2}^{\omega(K)} p_i^{\alpha_{l,i}+\alpha_{K,i}+\alpha_{f_3,i}+\alpha_{f_4,i}}\cdot\prod_{i=\omega(K)+1}^{\omega(f_3)} p_{i,f_3}^{\alpha_{f_3,i}}\cdot\prod_{i=\omega(K)+1}^{\omega(f_4)} p_{i,f_4}^{\alpha_{f_4,i}} +$$

$$\left.+2^{2\alpha_{l,1}+2\alpha_{K,1}}\cdot\prod_{i=2}^{\omega(K)} p_i^{2\alpha_{l,i}+2\alpha_{K,i}}\cdot\prod_{i=\omega(K)+1}^{\omega(f_4)} p_{i,f_4}^{\alpha_{f_4,i}}\right\} \in \mathbb{N}^+$$

(99)

A sufficient condition for (99) to be a positive natural number is

$$\alpha_{l,1} - 2\alpha_{f_4,1} - 4 +$$
$$+\min\left\{\alpha_{f_2,1}+\alpha_{f_4,1}+3, \alpha_{l,1}+\alpha_{K,1}+\alpha_{f_3,1}+2, 2\alpha_{l,1}+2\alpha_{K,1}\right\} \geq 0$$
$$\Leftrightarrow \alpha_{l,1} \geq 2\alpha_{f_4,1} + 4 -$$
$$-\min\left\{\alpha_{f_2,1}+\alpha_{f_4,1}+3, \alpha_{l,1}+\alpha_{K,1}+\alpha_{f_3,1}+2, 2\alpha_{l,1}+2\alpha_{K,1}\right\} \geq 0 \quad (100)$$

$$\alpha_{l,i} - 2\alpha_{f_4,i} + \min\left\{\alpha_{f_2,i}+\alpha_{f_4,i}, \alpha_{l,i}+\alpha_{K,i}+\alpha_{f_3,i}+\alpha_{f_4,i}, 2\alpha_{l,i}+2\alpha_{K,i}\right\} \geq 0$$

$$\Leftrightarrow \alpha_{l,i} \geq 2\alpha_{f_4,i} - \min\left\{\alpha_{f_2,i}+\alpha_{f_4,i}, \alpha_{l,i}+\alpha_{K,i}+\alpha_{f_3,i}+\alpha_{f_4,i}, 2\alpha_{l,i}+2\alpha_{K,i}\right\} \quad (101a)$$

for $i = 2,3,\ldots,\omega(K)$, if $p_2 \neq 3$.

If $p_2 = 3$, then for $i = 2$, (101a) becomes

$$\alpha_{l,2} \geq 2\alpha_{f_4,2} - \min\left\{\alpha_{f_2,2}+\alpha_{f_4,2}, \alpha_{l,2}+\alpha_{K,2}+\alpha_{f_3,2}+\alpha_{f_4,2}+1, 2\alpha_{l,2}+2\alpha_{K,2}\right\} \quad (101b)$$

In conclusion, rejoining the conditions (90)-(92), (94)-(95), (97)-(98a,b) and (100)-(101a,b), we have

$$\alpha_{f_4,1} + 4 -$$
$$-\min\left\{\begin{array}{l}\alpha_{K,1}+2,\\ \dfrac{\alpha_{K,1}-\alpha_{f_4,1}+\min\left\{\alpha_{f_2,1}+\alpha_{f_4,1}+4, \alpha_{l,1}+\alpha_{K,1}+\alpha_{f_3,1}+2, 2\alpha_{l,1}+2\alpha_{K,1}\right\}}{2},\\ 1+\min\left\{\alpha_{f_3,1}+1, \alpha_{l,1}+\alpha_{K,1}\right\},\\ -\alpha_{f_4,1}+\min\left\{\alpha_{f_2,1}+\alpha_{f_4,1}+3, \alpha_{l,1}+\alpha_{K,1}+\alpha_{f_3,1}+2, 2\alpha_{l,1}+2\alpha_{K,1}\right\}\end{array}\right\} \leq$$
$$\leq \alpha_{l,1} \leq \alpha_{f_4,1} + 2$$

(102)

$$\alpha_{f_4,i} - \min\left\{\begin{array}{l}\alpha_{K,i}, \\ \dfrac{\alpha_{K,i} - \alpha_{f_4,i} + \min\left\{\alpha_{f_2,i} + \alpha_{f_4,i}, \alpha_{l,i} + \alpha_{K,i} + \alpha_{f_3,i}, 2\alpha_{l,i} + 2\alpha_{K,i}\right\}}{2}, \\ \min\left\{\alpha_{f_3,i}, \alpha_{l,i} + \alpha_{K,i}\right\}, \\ -\alpha_{f_4,i} + \min\left\{\alpha_{f_2,i} + \alpha_{f_4,i}, \alpha_{l,i} + \alpha_{K,i} + \alpha_{f_3,i} + \alpha_{f_4,i}, 2\alpha_{l,i} + 2\alpha_{K,i}\right\}\end{array}\right\} \le$$

$$\le \alpha_{l,i} \le \alpha_{f_4,i}$$

(103a)

for $i = 2, 3, \ldots, \omega(K)$, when $p_2 \ne 3$. For $p_2 = 3$,

$$\alpha_{f_4,2} - \min\left\{\begin{array}{l}\alpha_{K,2}, \\ \dfrac{\alpha_{K,2} - \alpha_{f_4,2} + \min\left\{\alpha_{f_2,2} + \alpha_{f_4,2}, \alpha_{l,2} + \alpha_{K,2} + \alpha_{f_3,2}, 2\alpha_{l,2} + 2\alpha_{K,2}\right\}}{2}, \\ 1 + \min\left\{\alpha_{f_3,2}, \alpha_{l,2} + \alpha_{K,2}\right\}, \\ -\alpha_{f_4,2} + \min\left\{\alpha_{f_2,2} + \alpha_{f_4,2}, \alpha_{l,2} + \alpha_{K,2} + \alpha_{f_3,2} + \alpha_{f_4,2} + 1, 2\alpha_{l,2} + 2\alpha_{K,2}\right\}\end{array}\right\} \le$$

$$\le \alpha_{l,2} \le \alpha_{f_4,2}$$

(103b)

and

$$\alpha_{l,i} = \alpha_{f_4,i}, \qquad (104)$$

for $i = \omega(K) + 1, \ldots, \omega(f_4)$.

We mention that if the left hand side of the double inequalities (61)-(62) and (65)-(67) is negative, it is considered to be equal to 0.

Obviously, for $\alpha_{l,1} = \alpha_{f_4,1} + 2$ and $\alpha_{l,i} = \alpha_{f_4,i}$, $i = 2, 3, \ldots, \omega(K)$, it results that $Q = K$. To prove that there are also valid values of $Q$ smaller than K, it is sufficient to show that $\alpha_{l,i} = \alpha_{f_4,i} - 1$ verifies the first inequality in (103). Indeed, because for $i = 2, 3, \ldots, \omega(K)$, $\alpha_{f_2,i} \ge 1$, $\alpha_{f_3,i} \ge 1$, $\alpha_{f_4,i} \ge 1$, $\alpha_{K,i} \ge 1$, by replacing $\alpha_{l,i} = \alpha_{f_4,i} - 1$ into the left hand side of (103), we have

$$\alpha_{f_4,i} - \min\left\{\begin{array}{l}\alpha_{K,i}, \\ \dfrac{\alpha_{K,i} - \alpha_{f_4,i} + \min\left\{\alpha_{f_2,i} + \alpha_{f_4,i}, \alpha_{f_4,i} - 1 + \alpha_{K,i} + \alpha_{f_3,i}, 2\alpha_{f_4,i} - 2 + 2\alpha_{K,i}\right\}}{2}, \\ \min\left\{\alpha_{f_3,i}, \alpha_{f_4,i} - 1 + \alpha_{K,i}\right\}, \\ -\alpha_{f_4,i} + \min\left\{\alpha_{f_2,i} + \alpha_{f_4,i}, \alpha_{f_4,i} - 1 + \alpha_{K,i} + \alpha_{f_3,i} + \alpha_{f_4,i}, 2\alpha_{f_4,i} - 2 + 2\alpha_{K,i}\right\}\end{array}\right\} \le$$

$$\le \alpha_{f_4,i} - 1$$

(105)

Therefore, for any 4-PP satisfying constions stated in theorem 1 from [2], there is an ARP equivalent to the value of $Q < K$.

<u>Example 5</u>:

Let $K = 1696 = 2^5 \cdot 53^1$ and the 4-PP coefficients $f_1 = 55 = 5^1 \cdot 11^1$, $f_2 = 954 = 2^1 \cdot 53^1 \cdot 3^2$, $f_3 = 1272 = 2^3 \cdot 53^1 \cdot 3^1$, $f_4 = 1484 = 2^2 \cdot 53^1 \cdot 7^1$. These coefficients respect conditions a), c) and d) from theorem 1 in [2], resulting a valid 4-PP.

The values of $l$, leading to valid values of $Q$, are factorized as:

$$l = 2^{\alpha_{l,1}} \cdot 53^{\alpha_{l,2}} \cdot 7^{\alpha_{l,3}}, \qquad (106)$$

so that the factorization of $Q$ is:

$$Q = 2^{\alpha_{l,1}+1} \cdot 53^{\alpha_{l,2}} \cdot 7^{\alpha_{l,3}-1}. \qquad (107)$$

Condition (102) can be written as:

$$2+4-\min\left\{\begin{array}{l} 5+2, \\ \dfrac{5-2+\min\{1+2+4, \alpha_{l,1}+5+3+2, 2\alpha_{l,1}+10\}}{2}, \\ 1+\min\{3+1, \alpha_{l,1}+5\}, \\ -2+\min\{1+2+3, \alpha_{l,1}+5+3+2, 2\alpha_{l,1}+10\} \end{array}\right\} \le$$

$$\le \alpha_{l,1} \le 2+2$$

$$\Leftrightarrow 6-\min\left\{\begin{array}{l} 7, \\ \dfrac{3+\min\{7, \alpha_{l,1}+10, 2\alpha_{l,1}+10\}}{2}, \\ 1+\min\{4, \alpha_{l,1}+5\}, \\ -2+\min\{6, \alpha_{l,1}+10, 2\alpha_{l,1}+10\} \end{array}\right\} \le \alpha_{l,1} \le 4$$

$$\Leftrightarrow 6-\min\{7,5,5,4\} \le \alpha_{l,1} \le 4 \Leftrightarrow 2 \le \alpha_{l,1} \le 4. \qquad (108)$$

Condition (103a) can be written as

$$1-\min\left\{\begin{array}{l} 1, \\ \dfrac{1-1+\min\{1+1, \alpha_{l,2}+1+1, 2\alpha_{l,2}+2\}}{2}, \\ \min\{1, \alpha_{l,2}+1\}, \\ -1+\min\{1+1, \alpha_{l,2}+1+1+1, 2\alpha_{l,2}+2\} \end{array}\right\} \le \alpha_{l,2} \le 1$$

$$\Leftrightarrow 1-\min\left\{\begin{array}{l} 1, \\ \dfrac{\min\{2, \alpha_{l,2}+2, 2\alpha_{l,2}+2\}}{2}, \\ \min\{1, \alpha_{l,2}+1\}, \\ -1+\min\{2, \alpha_{l,2}+3, 2\alpha_{l,2}+2\} \end{array}\right\} \le \alpha_{l,2} \le 1$$

$$\Leftrightarrow 1-\min\{1,1,1,1\} \le \alpha_{l,2} \le 1 \Leftrightarrow 0 \le \alpha_{l,2} \le 1. \qquad (109)$$

From (104), we have

$$\alpha_{l,3} = 1. \qquad (110)$$

From (108)-(110), we see that there are 6 possible values for $l$ and, consequently, for $Q$, given in Table 7. The possible values for the vector $S$ for this interleaver, for which $Q_s = 8$ (the minimum value of $Q$) are given in Table 8.

**Table 7**. Possible values for $Q$ in the CPP with $K=1696$, $f_1=55$, $f_2=954$, $f_3=1272$ and $f_4=1484$.

| $l$ | 28 | 56 | 112 | 1484 | 2968 | 5936 |
|---|---|---|---|---|---|---|
| $Q$ | 8 | 16 | 32 | 424 | 848 | 1696 |

**Table 8**. Equivalent ARP interleavers with $P=f_1$, $Q_s=8$, $S(0)=S(4)=0$.

| $K$ | $f_1$ | $f_2$ | $f_3$ | $f_4$ | $S(1)$ | $S(2)$ | $S(3)$ | $S(5)$ | $S(6)$ | $S(7)$ |
|---|---|---|---|---|---|---|---|---|---|---|
| 1696 | 55 | 954 | 1272 | 1484 | 318 | 424 | 318 | 1166 | 424 | 1166 |

Example 6:

Let $K=216=2^3\cdot 3^3$ and the 4-PP coefficients $f_1=11$, $f_2=36=2^2\cdot 3^2$, $f_3=144=2^4\cdot 3^2$ și $f_4=162=2^1\cdot 3^4$.These coefficients respect conditions a), c) and d) from theorem 1 in [2], resulting in a valid 4-PP.

The values of $l$, leading to valid values of $Q$, are factorized as:

$$l=2^{\alpha_{l,1}}\cdot 3^{\alpha_{l,2}}, \qquad (111)$$

so that the factorization of $Q$ is

the $$Q=2^{\alpha_{l,1}}\cdot 3^{\alpha_{l,2}-1}. \qquad (112)$$

Condition (102) can be written as

$$1+4-\min\left\{\begin{matrix}3+2,\\ \dfrac{3-1+\min\{2+1+4,\alpha_{l,1}+3+4+2,2\alpha_{l,1}+6\}}{2},\\ 1+\min\{4+1,\alpha_{l,1}+3\},\\ -1+\min\{2+1+3,\alpha_{l,1}+3+4+2,2\alpha_{l,1}+6\}\end{matrix}\right\}\leq$$

$$\leq\alpha_{l,1}\leq 1+2$$

$$\Leftrightarrow 5-\min\left\{\begin{matrix}5,\\ \dfrac{2+\min\{7,\alpha_{l,1}+9,2\alpha_{l,1}+6\}}{2},\\ 1+\min\{5,\alpha_{l,1}+3\},\\ -1+\min\{6,\alpha_{l,1}+9,2\alpha_{l,1}+6\}\end{matrix}\right\}\leq\alpha_{l,1}\leq 3$$

$$\Leftrightarrow 5-\min\{5,4,4,5\}\leq\alpha_{l,1}\leq 3\Leftrightarrow 1\leq\alpha_{l,1}\leq 3. \qquad (113)$$

Condition (103b) can be written as

$$4-\min\left\{\begin{matrix}3,\\ \dfrac{3-4+\min\{2+4,\alpha_{l,2}+3+2,2\alpha_{l,2}+6\}}{2},\\ 1+\min\{2,\alpha_{l,2}+3\},\\ -4+\min\{2+4,\alpha_{l,2}+3+2+4+1,2\alpha_{l,2}+6\}\end{matrix}\right\}\leq\alpha_{l,2}\leq 4$$

$$\Leftrightarrow 4-\min\left\{\begin{array}{l}3,\\ \dfrac{-1+\min\{6,\alpha_{l,2}+5,2\alpha_{l,2}+6\}}{2},\\ 1+\min\{2,\alpha_{l,2}+3\},\\ -4+\min\{6,\alpha_{l,2}+10,2\alpha_{l,2}+6\}\end{array}\right\}\leq\alpha_{l,2}\leq 4$$

$$\Leftrightarrow 4-\min\{3,2,3,2\}\leq\alpha_{l,2}\leq 4 \Leftrightarrow 2\leq\alpha_{l,2}\leq 4\,. \qquad (114)$$

From (113)-(114), we see that there are 9 possible values for $l$ and, consequently, for $Q$, given in Table 9. The possible values for the vector $S$ for this interleaver, for which $Q_s=6$ (the minimum value of $Q$) are given in Table 10.

**Table 9**. Possible values for $Q$ in the CPP with $K=216$, $f_1=11$, $f_2=36$, $f_3=144$ and $f_4=162$.

| $l$ | 18 | 36 | 54 | 72 | 108 | 162 | 216 | 324 | 648 |
|---|---|---|---|---|---|---|---|---|---|
| $Q$ | 6 | 12 | 18 | 24 | 36 | 54 | 72 | 108 | 216 |

**Table 10**. Equivalent ARP interleavers with $P=f_1$, $Q_s=6$, $S(0)=S(2)=0$.

| $K$ | $f_1$ | $f_2$ | $f_3$ | $f_4$ | $S(1)$ | $S(3)$ | $S(4)$ | $S(5)$ |
|---|---|---|---|---|---|---|---|---|
| 216 | 11 | 36 | 144 | 162 | 126 | 54 | 72 | 54 |

Example 7:

Let $K=4802=2^1\cdot 7^4$ and the 4-PP coefficients $f_1=11$, $f_2=36=2^2\cdot 3^2$, $f_3=144=2^4\cdot 3^2$ și $f_4=162=2^1\cdot 3^4$.These coefficients respect conditions a), c) and d) from theorem 1 in [2], resulting in a valid 4-PP.

The values of $l$, leading to valid values of $Q$, are factorized as:

$$l=2^{\alpha_{l,1}}\cdot 7^{\alpha_{l,2}}, \qquad (115)$$

so that the factorization of $Q$ is

$$Q=2^{\alpha_{l,1}-2}\cdot 7^{\alpha_{l,2}+3}. \qquad (116)$$

Condition (102) can be written as

$$1+4-\min\left\{\begin{array}{l}1+2,\\ \dfrac{1-1+\min\{1+1+4,\alpha_{l,1}+1+0+2,2\alpha_{l,1}+2\}}{2},\\ 1+\min\{0+1,\alpha_{l,1}+1\},\\ -1+\min\{1+1+3,\alpha_{l,1}+1+0+2,2\alpha_{l,1}+2\}\end{array}\right\}\leq\alpha_{l,1}\leq 1+2$$

$$\Leftrightarrow 5-\min\left\{\begin{array}{l}3,\\ \dfrac{\min\{6,\alpha_{l,1}+3,2\alpha_{l,1}+2\}}{2},\\ 2,\\ -1+\min\{5,\alpha_{l,1}+3,2\alpha_{l,1}+2\}\end{array}\right\}\leq\alpha_{l,1}\leq 3\,. \qquad (117)$$

Because

$$\min\left\{\begin{array}{l}3,\\ \dfrac{\min\{6,\alpha_{l,1}+3,2\alpha_{l,1}+2\}}{2},\\ 2,\\ -1+\min\{5,\alpha_{l,1}+3,2\alpha_{l,1}+2\}\end{array}\right\}\le 2,$$

The double inequality in (117) is fulfilled only for

$\alpha_{l,1}=3$. (118)

Condition (103a) can be written as

$$1-\min\left\{\begin{array}{l}4,\\ \dfrac{4-1+\min\{1+1,\alpha_{l,2}+4+1,2\alpha_{l,2}+8\}}{2},\\ \min\{1,\alpha_{l,2}+1\},\\ -1+\min\{1+1,\alpha_{l,2}+4+1+1,2\alpha_{l,2}+8\}\end{array}\right\}\le\alpha_{l,2}\le 1$$

$$\Leftrightarrow 1-\min\left\{\begin{array}{l}4,\\ \dfrac{3+\min\{2,\alpha_{l,2}+5,2\alpha_{l,2}+8\}}{2},\\ \min\{1,\alpha_{l,2}+1\},\\ -1+\min\{2,\alpha_{l,2}+6,2\alpha_{l,2}+8\}\end{array}\right\}\le\alpha_{l,2}\le 1$$

$$\Leftrightarrow 1-\min\left\{4,\frac{3+2}{2},1,-1+2\right\}\le\alpha_{l,2}\le 1 \Leftrightarrow 0\le\alpha_{l,2}\le 1. \quad (119)$$

From (118)-(119), we see that there are 2 possible values for $l$ and, consequently, for $Q$, given in Table 11. As the minimum value of $Q$ is too large, namely $Q_s=686$, we have not given the possible values for the vector $S$ for this interleaver.

**Table 11**. Possible values for $Q$ in the 4-PP with $K=4802$, $f_1=10$, $f_2=70$, $f_3=7$ and $f_4=14$.

| $l$ | 8 | 56 |
|---|---|---|
| $Q$ | 686 | 4802 |

<u>Example 8:</u>

Let $K=1225=5^2\cdot 7^2$ and the 4-PP coefficients $f_1=12=2^2\cdot 3^1$, $f_2=525=3^1\cdot 5^2\cdot 7^1$, $f_3=140=2^2\cdot 5^1\cdot 7^1$ and $f_4=245=5^1\cdot 7^2$. These coefficients respect conditions a) and c) from theorem 1 in [2], resulting in a valid 4-PP.

The values of $l$, leading to valid values of $Q$, are factorized as:

$l=2^{\alpha_{l,1}}\cdot 5^{\alpha_{l,2}}\cdot 7^{\alpha_{l,3}}$, (120)

so that the factorization of $Q$ is:

$Q=2^{\alpha_{l,1}-2}\cdot 5^{\alpha_{l,2}+1}\cdot 7^{\alpha_{l,3}}$. (121)

In the following, we will consider $\alpha_{K,1}=0$, $\alpha_{f_2,1}=0$ şi $\alpha_{f_4,1}=0$.

Condition (102) can be written

$$0+4-\min\left\{\begin{array}{l}0+2,\\ \dfrac{0-0+\min\left\{0+0+4,\alpha_{l,1}+0+2+2,2\alpha_{l,1}+0\right\}}{2},\\ 1+\min\left\{2+1,\alpha_{l,1}+0\right\},\\ -0+\min\left\{0+0+3,\alpha_{l,1}+0+2+2,2\alpha_{l,1}+0\right\}\end{array}\right\}\le\alpha_{l,1}\le 0+2$$

$$\Leftrightarrow 4-\min\left\{\begin{array}{l}2,\\ \dfrac{\min\left\{4,\alpha_{l,1}+4,2\alpha_{l,1}\right\}}{2},\\ 1+\min\left\{3,\alpha_{l,1}\right\},\\ \min\left\{3,\alpha_{l,1}+4,2\alpha_{l,1}\right\}\end{array}\right\}\le\alpha_{l,1}\le 2\,. \qquad (122)$$

Because

$$\min\left\{\begin{array}{l}2,\\ \dfrac{\min\left\{4,\alpha_{l,1}+4,2\alpha_{l,1}\right\}}{2},\\ 1+\min\left\{3,\alpha_{l,1}\right\},\\ \min\left\{3,\alpha_{l,1}+4,2\alpha_{l,1}\right\}\end{array}\right\}\le 2\,,$$

The double inequality in (122) is fulfilled only for

$$\alpha_{l,1}=2\,. \qquad (123)$$

Condition (103a) can be written as

$$\alpha_{f_4,2}-\min\left\{\begin{array}{l}\alpha_{K,2},\\ \dfrac{\alpha_{K,2}-\alpha_{f_4,2}+\min\left\{\alpha_{f_2,2}+\alpha_{f_4,2},\alpha_{l,2}+\alpha_{K,2}+\alpha_{f_3,2},2\alpha_{l,2}+2\alpha_{K,2}\right\}}{2},\\ \min\left\{\alpha_{f_3,2},\alpha_{l,2}+\alpha_{K,2}\right\},\\ -\alpha_{f_4,2}+1+\min\left\{\alpha_{f_2,2}+\alpha_{f_4,2},\alpha_{l,2}+\alpha_{K,2}+\alpha_{f_3,2},2\alpha_{l,2}+2\alpha_{K,2}\right\}\end{array}\right\}\le$$

$$\le\alpha_{l,2}\le\alpha_{f_4,2}$$

$$1-\min\left\{\begin{array}{l}2,\\ \dfrac{2-1+\min\left\{2+1,\alpha_{l,2}+2+1,2\alpha_{l,2}+4\right\}}{2},\\ \min\left\{1,\alpha_{l,2}+2\right\},\\ -1+\min\left\{2+1,\alpha_{l,2}+2+1+1,2\alpha_{l,2}+4\right\}\end{array}\right\}\le\alpha_{l,2}\le 1$$

$$\Leftrightarrow 1-\min\left\{\begin{array}{l}2,\\ \dfrac{1+\min\left\{3,\alpha_{l,2}+3,2\alpha_{l,2}+4\right\}}{2},\\ \min\left\{1,\alpha_{l,2}+2\right\},\\ -1+\min\left\{3,\alpha_{l,2}+4,2\alpha_{l,2}+4\right\}\end{array}\right\}\le\alpha_{l,2}\le 1$$

$$\Leftrightarrow 1-\min\left\{2,\frac{1+3}{2},1,-1+3\right\}\le\alpha_{l,2}\le 1\Leftrightarrow 0\le\alpha_{l,2}\le 1 \qquad (124)$$

$$2-\min\left\{\begin{array}{l}2,\\ \dfrac{2-2+\min\left\{1+2,\alpha_{l,3}+2+1,2\alpha_{l,3}+4\right\}}{2},\\ \min\left\{1,\alpha_{l,3}+2\right\},\\ -2+\min\left\{1+2,\alpha_{l,3}+2+1+2,2\alpha_{l,3}+4\right\}\end{array}\right\}\le\alpha_{l,3}\le 2$$

$$\Leftrightarrow 2-\min\left\{\begin{array}{l}2,\\ \dfrac{\min\left\{3,\alpha_{l,3}+3,2\alpha_{l,3}+4\right\}}{2},\\ \min\left\{1,\alpha_{l,3}+2\right\},\\ -2+\min\left\{3,\alpha_{l,3}+5,2\alpha_{l,3}+4\right\}\end{array}\right\}\le\alpha_{l,3}\le 2$$

$$\Leftrightarrow 2-\min\left\{2,\frac{3}{2},1,-2+3\right\}\le\alpha_{l,3}\le 2\Leftrightarrow 1\le\alpha_{l,3}\le 2. \tag{125}$$

From (123)-(125), we see that there are 4 possible values for $l$ and, consequently, for $Q$, given in Table 12. As the minimum value of $Q$ is too large, namely $Q_s = 35$, we have not given the possible values for the vector $S$ for this interleaver.

**Table 12**. Possible values for $Q$ in the CPP with $K = 1225$, $f_1 = 12$, $f_2 = 525$, $f_3 = 140$ and $f_4 = 245$.

| $l$ | 28 | 140 | 196 | 980 |
|---|---|---|---|---|
| $Q$ | 35 | 175 | 245 | 1225 |

## 4.4. 5-PP interleavers seen as peculiar cases of ARP interleavers

From (27), we have:

$$Q=\frac{l\cdot K}{5\cdot f_5}=2^{\alpha_{l,1}+\alpha_{K,1}-\alpha_{f_5,1}}\cdot 5^{-1}\cdot\prod_{i=2}^{\omega(K)}p_i^{\alpha_{l,i}+\alpha_{K,i}-\alpha_{f_5,i}}\cdot\prod_{i=\omega(K)+1}^{\omega(f_5)}p_{i,f_4}^{\alpha_{l,i}-\alpha_{f_5,i}} \tag{126}$$

In this case, we consider the factorization of $K$, of the coefficients $f_2$, $f_3$, $f_4$, $f_5$, and of $l$ depending on whether $5\nmid K$ or not.

Thus, if $5\nmid K$, the quantities $K$, $f_2$, $f_3$ şi $f_4$ are the same as in (32) and (33), respectively, and

$$f_5=2^{\alpha_{f_5,1}}\cdot\prod_{i=2}^{\omega(K)}p_i^{\alpha_{f_5,i}}\cdot 5^{\alpha_{f_5,\omega(K)+1}}\cdot\prod_{i=\omega(K)+2}^{\omega(f_5)}p_{i,f_5}^{\alpha_{f_5,i}} \tag{127}$$

$$l=2^{\alpha_{l,1}}\cdot\prod_{i=2}^{\omega(K)}p_i^{\alpha_{l,i}}\cdot 5^{\alpha_{l,\omega(K)+1}}\cdot\prod_{i=\omega(K)+2}^{\omega(f_5)}p_{i,f_5}^{\alpha_{l,i}} \tag{128}$$

If $5\mid K$, the quantities $f_2$, $f_3$ şi $f_4$ are the same as in (33), and

$$K=2^{\alpha_{K,1}}\cdot 5^{\alpha_{K,2}}\cdot\prod_{i=3}^{\omega(K)}p_i^{\alpha_{K,i}} \tag{129}$$

$$f_5=2^{\alpha_{f_5,1}}\cdot 5^{\alpha_{f_5,2}}\cdot\prod_{i=3}^{\omega(K)}p_i^{\alpha_{f_5,i}}\cdot\prod_{i=\omega(K)+1}^{\omega(f_5)}p_{i,f_5}^{\alpha_{f_5,i}} \tag{130}$$

$$l = 2^{\alpha_{l,1}} \cdot 5^{\alpha_{l,2}} \cdot \prod_{i=3}^{\omega(K)} p_i^{\alpha_{l,i}} \cdot \prod_{i=\omega(K)+1}^{\omega(f_5)} p_{i,f_5}^{\alpha_{l,i}} \tag{131}$$

We mention that in (32)-(33) it could be $\alpha_{K,1} = 0$ and/or $\alpha_{f_2,1} = 0$ and/or $\alpha_{f_3,1} = 0$ and/or $\alpha_{f_4,1} = 0$ and/or $\alpha_{f_5,1} = 0$ and/or $\alpha_{f_5,\omega(K)+1} = 0$, the other powers of the prime numbers being greater than 0.

Then, if $5 \nmid K$, we have

$$Q = 2^{\alpha_{l,1}+\alpha_{K,1}-\alpha_{f_5,1}} \cdot \prod_{i=2}^{\omega(K)} p_i^{\alpha_{l,i}+\alpha_{K,i}-\alpha_{f_5,i}} \cdot 5^{\alpha_{l,\omega(K)+1}-\alpha_{f_5,\omega(K)+1}-1} \cdot \prod_{i=\omega(K)+2}^{\omega(f_5)} p_{i,f_5}^{\alpha_{l,i}-\alpha_{f_5,i}} \tag{132}$$

and if $5 | K$

$$Q = 2^{\alpha_{l,1}+\alpha_{K,1}-\alpha_{f_5,1}} \cdot 5^{\alpha_{l,2}-\alpha_{f_5,1}-1} \cdot \prod_{i=3}^{\omega(K)} p_i^{\alpha_{l,i}+\alpha_{K,i}-\alpha_{f_5,i}} \cdot \prod_{i=\omega(K)+1}^{\omega(f_5)} p_{i,f_5}^{\alpha_{l,i}-\alpha_{f_5,i}} \tag{133}$$

As $Q$ has to be a divisor of *K*, from (132) we have:

I) If $5 \nmid K$,

$$0 \le \alpha_{l,i} + \alpha_{K,i} - \alpha_{f_5,i} \le \alpha_{K,i} \Leftrightarrow \alpha_{f_5,i} - \alpha_{K,i} \le \alpha_{l,i} \le \alpha_{f_5,i} \tag{134}$$

for $i = 1,2,3,\ldots,\omega(K)$,

$$\alpha_{l,\omega(K)+1} = \alpha_{f_5,\omega(K)+1} + 1 \tag{135}$$

and

$$\alpha_{l,i} = \alpha_{f_5,i}, \tag{136}$$

for $i = \omega(K)+2,\ldots,\omega(f_5)$.

II) If $5 | K$,

$$0 \le \alpha_{l,i} + \alpha_{K,i} - \alpha_{f_5,i} \le \alpha_{K,i} \Leftrightarrow \alpha_{f_5,i} - \alpha_{K,i} \le \alpha_{l,i} \le \alpha_{f_5,i} \tag{137}$$

for $i = 1,3,\ldots,\omega(K)$,

$$0 \le \alpha_{l,2} + \alpha_{K,2} - \alpha_{f_5,2} - 1 \le \alpha_{K,2} \Leftrightarrow \alpha_{f_5,2} - \alpha_{K,2} + 1 \le \alpha_{l,2} \le \alpha_{f_5,2} + 1 \tag{138}$$

and

$$\alpha_{l,i} = \alpha_{f_5,i}, \tag{139}$$

for $i = \omega(K)+1,\ldots,\omega(f_5)$.

From (28), we have:

I) If $5 \nmid K$, considering conditions (135)-(136), we have

$$\frac{l^2 \cdot K \cdot \left(5^3 \cdot f_2 \cdot f_5^2 + 5^2 \cdot f_3 \cdot f_5 \cdot l \cdot K + 5 \cdot f_4 \cdot l^2 \cdot K^2 + l^3 \cdot K^3\right)}{5^5 \cdot f_5^4} = 2^{2\alpha_{l,1}+\alpha_{K,1}+\alpha_{f_2,1}-2\alpha_{f_5,1}} \cdot$$

$$\cdot \prod_{i=2}^{\omega(K)} p_i^{2\alpha_{l,i}+\alpha_{K,i}+\alpha_{f_2,i}-2\alpha_{f_5,i}} \cdot \prod_{i=\omega(K)+1}^{\omega(f_2)} p_{i,f_2}^{\alpha_{f_2,i}} \cdot 5^{2\alpha_{l,\omega(K)+1}-2\alpha_{f_5,\omega(K)+1}-2} \cdot \prod_{i=\omega(K)+2}^{\omega(f_5)} p_{i,f_5}^{2\alpha_{l,i}-2\alpha_{f_5,i}} +$$

$$+2^{3\alpha_{l,1}+2\alpha_{K,1}+\alpha_{f_3,1}-3\alpha_{f_5,1}} \cdot \prod_{i=2}^{\omega(K)} p_i^{3\alpha_{l,i}+2\alpha_{K,i}+\alpha_{f_3,i}-3\alpha_{f_5,i}} \cdot \prod_{i=\omega(K)+1}^{\omega(f_3)} p_{i,f_3}^{\alpha_{f_3,i}} \cdot 5^{3\alpha_{l,\omega(K)+1}-3\alpha_{f_5,\omega(K)+1}-3} \cdot$$

$$\cdot \prod_{i=\omega(K)+2}^{\omega(f_5)} p_{i,f_5}^{3\alpha_{l,i}-3\alpha_{f_5,i}} + 2^{4\alpha_{l,1}+3\alpha_{K,1}+\alpha_{f_4,1}-4\alpha_{f_5,1}} \cdot \prod_{i=2}^{\omega(K)} p_i^{4\alpha_{l,i}+3\alpha_{K,i}+\alpha_{f_4,i}-4\alpha_{f_5,i}} \cdot \prod_{i=\omega(K)+1}^{\omega(f_4)} p_{i,f_4}^{\alpha_{f_4,i}} \cdot$$

$$\cdot 5^{4\alpha_{l,\omega(K)+1}-4\alpha_{f_5,\omega(K)+1}-4} \cdot \prod_{i=\omega(K)+2}^{\omega(f_5)} p_{i,f_5}^{4\alpha_{l,i}-4\alpha_{f_5,i}} + 2^{5\alpha_{l,1}+4\alpha_{K,1}-4\alpha_{f_5,1}} \cdot \prod_{i=2}^{\omega(K)} p_i^{5\alpha_{l,i}+4\alpha_{K,i}-4\alpha_{f_5,i}} \cdot$$

$$\cdot 5^{5\alpha_{l,\omega(K)+1}-4\alpha_{f_5,\omega(K)+1}-5}\cdot\prod_{i=\omega(K)+2}^{\omega(f_5)}p_{i,f_5}^{5\alpha_{l,i}-4\alpha_{f_5,i}}=2^{2\alpha_{l,1}+\alpha_{K,1}-4\alpha_{f_5,1}}\cdot\prod_{i=2}^{\omega(K)}p_i^{2\alpha_{l,i}+\alpha_{K,i}-4\alpha_{f_5,i}}\cdot$$

$$\cdot\left\{2^{\alpha_{f_2,1}+2\alpha_{f_5,1}}\cdot\prod_{i=2}^{\omega(K)}p_i^{\alpha_{f_2,i}+2\alpha_{f_5,i}}\cdot\prod_{i=\omega(K)+1}^{\omega(f_2)}p_{i,f_2}^{\alpha_{f_2,i}}+2^{\alpha_{l,1}+\alpha_{K,1}+\alpha_{f_3,1}+\alpha_{f_5,1}}\cdot\prod_{i=2}^{\omega(K)}p_i^{\alpha_{l,i}+\alpha_{K,i}+\alpha_{f_3,i}+\alpha_{f_5,i}}\cdot\right.$$

$$\cdot\prod_{i=\omega(K)+1}^{\omega(f_3)}p_{i,f_3}^{\alpha_{f_3,i}}+2^{2\alpha_{l,1}+2\alpha_{K,1}+\alpha_{f_4,1}}\cdot\prod_{i=2}^{\omega(K)}p_i^{2\alpha_{l,i}+2\alpha_{K,i}+\alpha_{f_4,i}}\cdot\prod_{i=\omega(K)+1}^{\omega(f_4)}p_{i,f_4}^{\alpha_{f_4,i}}+$$

$$\left.+2^{3\alpha_{l,1}+3\alpha_{K,1}}\cdot\prod_{i=2}^{\omega(K)}p_i^{3\alpha_{l,i}+3\alpha_{K,i}}\cdot\prod_{i=\omega(K)+1}^{\omega(f_5)}p_{i,f_5}^{\alpha_{f_5,i}}\right\}\in\mathbb{N}^+\quad(140)$$

A sufficient condition for the expression in (140) to be a natural positive number is:

$$2\alpha_{l,i}+\alpha_{K,i}-4\alpha_{f_5,i}+$$

$$+\min\left\{\alpha_{f_2,i}+2\alpha_{f_5,i},\alpha_{l,i}+\alpha_{K,i}+\alpha_{f_3,i}+\alpha_{f_5,i},2\alpha_{l,i}+2\alpha_{K,i}+\alpha_{f_4,i},3\alpha_{l,i}+3\alpha_{K,i}\right\}\geq 0$$

$$\Leftrightarrow\alpha_{l,i}\geq\frac{4\alpha_{f_5,i}-\alpha_{K,i}}{2}-$$

$$-\frac{\min\left\{\alpha_{f_2,i}+2\alpha_{f_5,i},\alpha_{l,i}+\alpha_{K,i}+\alpha_{f_3,i}+\alpha_{f_5,i},2\alpha_{l,i}+2\alpha_{K,i}+\alpha_{f_4,i},3\alpha_{l,i}+3\alpha_{K,i}\right\}}{2}\quad(141)$$

for $i=1,2,3,\ldots,\omega(K)$.

II) If $5\,|\,K$, considering (139), we have

$$\frac{l^2\cdot K\cdot\left(5^3\cdot f_2\cdot f_5^2+5^2\cdot f_3\cdot f_5\cdot l\cdot K+5\cdot f_4\cdot l^2\cdot K^2+l^3\cdot K^3\right)}{5^5\cdot f_5^4}=2^{2\alpha_{l,1}+\alpha_{K,1}+\alpha_{f_2,1}-2\alpha_{f_5,1}}\cdot$$

$$\cdot 5^{2\alpha_{l,2}+\alpha_{K,2}+\alpha_{f_2,2}-2\alpha_{f_5,2}-2}\cdot\prod_{i=3}^{\omega(K)}p_i^{2\alpha_{l,i}+\alpha_{K,i}+\alpha_{f_2,i}-2\alpha_{f_5,i}}\cdot\prod_{i=\omega(K)+1}^{\omega(f_2)}p_{i,f_2}^{\alpha_{f_2,i}}\cdot\prod_{i=\omega(K)+1}^{\omega(f_5)}p_{i,f_5}^{2\alpha_{l,i}-2\alpha_{f_5,i}}+$$

$$+2^{3\alpha_{l,1}+2\alpha_{K,1}+\alpha_{f_3,1}-3\alpha_{f_5,1}}\cdot 5^{3\alpha_{l,2}+2\alpha_{K,2}+\alpha_{f_3,2}-3\alpha_{f_5,2}-3}\cdot\prod_{i=3}^{\omega(K)}p_i^{3\alpha_{l,i}+2\alpha_{K,i}+\alpha_{f_3,i}-3\alpha_{f_5,i}}\cdot\prod_{i=\omega(K)+1}^{\omega(f_3)}p_{i,f_3}^{\alpha_{f_3,i}}\cdot$$

$$\cdot\prod_{i=\omega(K)+1}^{\omega(f_5)}p_{i,f_5}^{3\alpha_{l,i}-3\alpha_{f_5,i}}+2^{4\alpha_{l,1}+3\alpha_{K,1}+\alpha_{f_4,1}-4\alpha_{f_5,1}}\cdot 5^{4\alpha_{l,2}+3\alpha_{K,2}+\alpha_{f_4,2}-4\alpha_{f_5,2}-4}\cdot\prod_{i=3}^{\omega(K)}p_i^{4\alpha_{l,i}+3\alpha_{K,i}+\alpha_{f_4,i}-4\alpha_{f_5,i}}\cdot$$

$$\cdot\prod_{i=\omega(K)+1}^{\omega(f_4)}p_{i,f_4}^{\alpha_{f_4,i}}\cdot\prod_{i=\omega(K)+1}^{\omega(f_5)}p_{i,f_5}^{4\alpha_{l,i}-4\alpha_{f_5,i}}+2^{5\alpha_{l,1}+4\alpha_{K,1}-4\alpha_{f_5,1}}\cdot 5^{5\alpha_{l,2}+4\alpha_{K,2}-4\alpha_{f_5,2}-5}\cdot\prod_{i=3}^{\omega(K)}p_i^{5\alpha_{l,i}+4\alpha_{K,i}-4\alpha_{f_5,i}}\cdot$$

$$\cdot\prod_{i=\omega(K)+1}^{\omega(f_5)}p_{i,f_5}^{5\alpha_{l,i}-4\alpha_{f_5,i}}=2^{2\alpha_{l,1}+\alpha_{K,1}-4\alpha_{f_5,1}}\cdot 5^{2\alpha_{l,2}+\alpha_{K,2}-4\alpha_{f_5,2}-5}\cdot\prod_{i=3}^{\omega(K)}p_i^{2\alpha_{l,i}+\alpha_{K,i}-4\alpha_{f_5,i}}\cdot$$

$$\cdot\left\{2^{\alpha_{f_2,1}+2\alpha_{f_5,1}}\cdot 5^{\alpha_{f_2,2}+2\alpha_{f_5,2}+3}\cdot\prod_{i=3}^{\omega(K)}p_i^{\alpha_{f_2,i}+2\alpha_{f_5,i}}\cdot\prod_{i=\omega(K)+1}^{\omega(f_2)}p_{i,f_2}^{\alpha_{f_2,i}}+2^{\alpha_{l,1}+\alpha_{K,1}+\alpha_{f_3,1}+\alpha_{f_5,1}}\cdot\right.$$

$$\cdot 5^{\alpha_{l,2}+\alpha_{K,2}+\alpha_{f_3,2}+\alpha_{f_5,2}+2}\cdot\prod_{i=3}^{\omega(K)}p_i^{\alpha_{l,i}+\alpha_{K,i}+\alpha_{f_3,i}+\alpha_{f_5,i}}\cdot\prod_{i=\omega(K)+1}^{\omega(f_3)}p_{i,f_3}^{\alpha_{f_3,i}}+2^{2\alpha_{l,1}+2\alpha_{K,1}+\alpha_{f_4,1}}\cdot 5^{2\alpha_{l,2}+2\alpha_{K,2}+\alpha_{f_4,2}+1}\cdot$$

$$\left.\cdot\prod_{i=3}^{\omega(K)}p_i^{2\alpha_{l,i}+2\alpha_{K,i}+\alpha_{f_4,i}}\cdot\prod_{i=\omega(K)+1}^{\omega(f_4)}p_{i,f_4}^{\alpha_{f_4,i}}++2^{3\alpha_{l,1}+3\alpha_{K,1}}\cdot 5^{3\alpha_{l,2}+3\alpha_{K,2}}\cdot\prod_{i=3}^{\omega(K)}p_i^{3\alpha_{l,i}+3\alpha_{K,i}}\cdot\prod_{i=\omega(K)+1}^{\omega(f_5)}p_{i,f_5}^{\alpha_{f_5,i}}\right\}\in\mathbb{N}^+$$

(142)

A sufficient condition for the expression in (131) to be a natural positive number is:

$$\alpha_{l,2} \geq \frac{4\alpha_{f_5,2} - \alpha_{K,2} + 5}{2} - \frac{\min\left\{\alpha_{f_2,2} + 2\alpha_{f_5,2} + 3, \alpha_{l,2} + \alpha_{K,2} + \alpha_{f_3,2} + \alpha_{f_5,2} + 2, 2\alpha_{l,2} + 2\alpha_{K,2} + \alpha_{f_4,2} + 1, 3\alpha_{l,2} + 3\alpha_{K,2}\right\}}{2}$$

(143)

and

$$\alpha_{l,i} \geq \frac{4\alpha_{f_5,i} - \alpha_{K,i}}{2} - \frac{\min\left\{\alpha_{f_2,i} + 2\alpha_{f_5,i}, \alpha_{l,i} + \alpha_{K,i} + \alpha_{f_3,i} + \alpha_{f_5,i}, 2\alpha_{l,i} + 2\alpha_{K,i} + \alpha_{f_4,i}, 3\alpha_{l,i} + 3\alpha_{K,i}\right\}}{2} \quad (144)$$

for $i = 1,3,\ldots,\omega(K)$.

From (29), we have:

I) If $5 \nmid K$, considering conditions (135)-(136), we have:

$$\frac{2\cdot l\cdot(2\cdot f_4 + l\cdot K)}{5\cdot f_5} = 2^{\alpha_{l,1}+\alpha_{f_4,1}-\alpha_{f_5,1}+2}\cdot\prod_{i=2}^{\omega(K)} p_i^{\alpha_{l,i}+\alpha_{f_4,i}-\alpha_{f_5,i}}\cdot\prod_{i=\omega(K)+1}^{\omega(f_4)} p_{i,f_4}^{\alpha_{f_4,i}}\cdot$$
$$\cdot 5^{\alpha_{l,\omega(K)+1}-\alpha_{f_5,\omega(K)+1}-1}\cdot\prod_{i=\omega(K)+2}^{\omega(f_5)} p_{i,f_5}^{\alpha_{l,i}-\alpha_{f_5,i}} + 2^{2\alpha_{l,1}+\alpha_{K,1}-\alpha_{f_5,1}+1}\cdot\prod_{i=2}^{\omega(K)} p_i^{2\alpha_{l,i}+\alpha_{K,i}-\alpha_{f_5,i}}\cdot$$
$$\cdot 5^{2\alpha_{l,\omega(K)+1}-\alpha_{f_5,\omega(K)+1}-1}\cdot\prod_{i=\omega(K)+2}^{\omega(f_5)} p_{i,f_5}^{2\alpha_{l,i}-\alpha_{f_5,i}} = 2^{\alpha_{l,1}-\alpha_{f_5,1}+1}\cdot\prod_{i=2}^{\omega(K)} p_i^{\alpha_{l,i}-\alpha_{f_5,i}}\cdot$$
$$\left\{2^{\alpha_{f_4,1}+1}\cdot\prod_{i=2}^{\omega(K)} p_i^{\alpha_{f_4,i}}\cdot\prod_{i=\omega(K)+1}^{\omega(f_4)} p_{i,f_4}^{\alpha_{f_4,i}} + 2^{\alpha_{l,1}+\alpha_{K,1}}\cdot\prod_{i=2}^{\omega(K)} p_i^{\alpha_{l,i}+\alpha_{K,i}}\cdot 5^{\alpha_{f_5,\omega(K)+1}+1}\cdot\prod_{i=\omega(K)+2}^{\omega(f_5)} p_{i,f_5}^{\alpha_{f_5,i}}\right\}\in\mathbb{N}^+$$

(145)

A sufficient condition for the expression in (145) to be a natural positive number is:

$$\alpha_{l,1} - \alpha_{f_5,1} + 1 + \min\left\{\alpha_{f_4,1} + 1, \alpha_{l,1} + \alpha_{K,1}\right\} \geq 0$$
$$\Leftrightarrow \alpha_{l,1} \geq \alpha_{f_5,1} - 1 - \min\left\{\alpha_{f_4,1} + 1, \alpha_{l,1} + \alpha_{K,1}\right\} \quad (146)$$
$$\alpha_{l,i} - \alpha_{f_5,i} + \min\left\{\alpha_{f_4,i}, \alpha_{l,i} + \alpha_{K,i}\right\} \geq 0 \;\Leftrightarrow \alpha_{l,i} \geq \alpha_{f_5,i} - \min\left\{\alpha_{f_4,i}, \alpha_{l,i} + \alpha_{K,i}\right\} \quad (147)$$

for $i = 2,3,\ldots,\omega(K)$.

II) If $5 \mid K$, considering (139), we have:

$$\frac{2\cdot l\cdot\left(2\cdot f_4+l\cdot K\right)}{5\cdot f_5}=2^{\alpha_{l,1}+\alpha_{f_4,1}-\alpha_{f_5,1}+2}\cdot 5^{\alpha_{l,2}+\alpha_{f_4,2}-\alpha_{f_5,2}-1}\cdot\prod_{i=3}^{\omega(K)}p_i^{\alpha_{l,i}+\alpha_{f_4,i}-\alpha_{f_5,i}}\cdot\prod_{i=\omega(K)+1}^{\omega(f_4)}p_{i,f_4}^{\alpha_{f_4,i}}\cdot$$

$$\cdot\prod_{i=\omega(K)+1}^{\omega(f_5)}p_{i,f_5}^{\alpha_{l,i}-\alpha_{f_5,i}}+2^{2\alpha_{l,1}+\alpha_{K,1}-\alpha_{f_5,1}+1}\cdot 5^{2\alpha_{l,2}+\alpha_{K,2}-\alpha_{f_5,2}-1}\cdot\prod_{i=3}^{\omega(K)}p_i^{2\alpha_{l,i}+\alpha_{K,i}-\alpha_{f_5,i}}\cdot$$

$$\cdot\prod_{i=\omega(K)+1}^{\omega(f_5)}p_{i,f_5}^{2\alpha_{l,i}-\alpha_{f_5,i}}=2^{\alpha_{l,1}-\alpha_{f_5,1}+1}\cdot 5^{\alpha_{l,2}-\alpha_{f_5,2}-1}\cdot\prod_{i=3}^{\omega(K)}p_i^{\alpha_{l,i}-\alpha_{f_5,i}}\cdot$$

$$\left\{2^{\alpha_{f_4,1}+1}\cdot 5^{\alpha_{f_4,2}}\cdot\prod_{i=3}^{\omega(K)}p_i^{\alpha_{f_4,i}}\cdot\prod_{i=\omega(K)+1}^{\omega(f_4)}p_{i,f_4}^{\alpha_{f_4,i}}+2^{\alpha_{l,1}+\alpha_{K,1}}\cdot 5^{\alpha_{l,2}+\alpha_{K,2}}\cdot\prod_{i=3}^{\omega(K)}p_i^{\alpha_{l,i}+\alpha_{K,i}}\cdot\prod_{i=\omega(K)+1}^{\omega(f_5)}p_{i,f_5}^{\alpha_{f_5,i}}\right\}\in\mathbb{N}^+$$

(148)

A sufficient condition for the expression in (148) to be a natural positive number is:

$$\alpha_{l,1}-\alpha_{f_5,1}+1+\min\left\{\alpha_{f_4,1}+1,\alpha_{l,1}+\alpha_{K,1}\right\}\geq 0$$

$$\Leftrightarrow\alpha_{l,1}\geq\alpha_{f_5,1}-1-\min\left\{\alpha_{f_4,1}+1,\alpha_{l,1}+\alpha_{K,1}\right\}\qquad(149)$$

$$\alpha_{l,2}-\alpha_{f_5,2}-1+\min\left\{\alpha_{f_4,2},\alpha_{l,2}+\alpha_{K,2}\right\}\geq 0$$

$$\Leftrightarrow\alpha_{l,2}\geq\alpha_{f_5,2}+1-\min\left\{\alpha_{f_4,2},\alpha_{l,2}+\alpha_{K,2}\right\}\qquad(150)$$

$$\alpha_{l,i}-\alpha_{f_5,i}+\min\left\{\alpha_{f_4,i},\alpha_{l,i}+\alpha_{K,i}\right\}\geq 0\ \Leftrightarrow\alpha_{l,i}\geq\alpha_{f_5,i}-\min\left\{\alpha_{f_4,i},\alpha_{l,i}+\alpha_{K,i}\right\}\quad(151)$$

for $i=3,\ldots,\omega(K)$.

From (30), we have:

I) If $5\nmid K$, considering conditions (135)-(136), we have:

$$\frac{l\cdot\left(5\cdot 3\cdot f_3\cdot f_5+5\cdot 2\cdot 3\cdot f_4\cdot f_5\cdot l\cdot K+2\cdot l^2\cdot K^2\right)}{5^2\cdot f_5^2}=2^{\alpha_{l,1}+\alpha_{f_3,1}-\alpha_{f_5,1}}\cdot 3\cdot\prod_{i=2}^{\omega(K)}p_i^{\alpha_{l,i}+\alpha_{f_3,i}-\alpha_{f_5,i}}\cdot$$

$$\cdot\prod_{i=\omega(K)+1}^{\omega(f_3)}p_{i,f_3}^{\alpha_{f_3,i}}\cdot 5^{\alpha_{l,\omega(K)+1}-\alpha_{f_5,\omega(K)+1}-1}\cdot\prod_{i=\omega(K)+2}^{\omega(f_5)}p_{i,f_5}^{\alpha_{l,i}-\alpha_{f_5,i}}+2^{2\alpha_{l,1}+\alpha_{K,1}+\alpha_{f_4,1}-\alpha_{f_5,1}+1}\cdot 3\cdot$$

$$\cdot\prod_{i=2}^{\omega(K)}p_i^{2\alpha_{l,i}+\alpha_{K,i}+\alpha_{f_4,i}-\alpha_{f_5,i}}\cdot\prod_{i=\omega(K)+1}^{\omega(f_4)}p_{i,f_4}^{\alpha_{f_4,i}}\cdot 5^{2\alpha_{l,\omega(K)+1}-\alpha_{f_5,\omega(K)+1}-1}\cdot\prod_{i=\omega(K)+2}^{\omega(f_5)}p_{i,f_5}^{2\alpha_{l,i}-\alpha_{f_5,i}}+$$

$$+2^{3\alpha_{l,1}+2\alpha_{K,1}-2\alpha_{f_5,1}+1}\cdot\prod_{i=2}^{\omega(K)}p_i^{3\alpha_{l,i}+2\alpha_{K,i}-2\alpha_{f_5,i}}\cdot 5^{3\alpha_{l,\omega(K)+1}-2\alpha_{f_5,\omega(K)+1}-2}\cdot\prod_{i=\omega(K)+2}^{\omega(f_5)}p_{i,f_5}^{3\alpha_{l,i}-2\alpha_{5,i}}=$$

$$=2^{\alpha_{l,1}-2\alpha_{f_5,1}}\cdot\prod_{i=2}^{\omega(K)}p_i^{\alpha_{l,i}-2\alpha_{f_5,i}}\cdot\left\{2^{\alpha_{f_3,1}+\alpha_{f_5,1}}\cdot 3\cdot\prod_{i=2}^{\omega(K)}p_i^{\alpha_{f_3,i}+\alpha_{f_5,i}}\cdot\prod_{i=\omega(K)+1}^{\omega(f_3)}p_{i,f_3}^{\alpha_{f_3,i}}+\right.$$

$$+2^{\alpha_{l,1}+\alpha_{K,1}+\alpha_{f_4,1}+\alpha_{f_5,1}+1}\cdot 3\cdot\prod_{i=2}^{\omega(K)}p_i^{\alpha_{l,i}+\alpha_{K,i}+\alpha_{f_4,i}+\alpha_{f_5,i}}\cdot\prod_{i=\omega(K)+1}^{\omega(f_4)}p_{i,f_4}^{\alpha_{f_4,i}}\cdot 5^{\alpha_{f_5,\omega(K)+1}+1}\cdot\prod_{i=\omega(K)+2}^{\omega(f_5)}p_{i,f_5}^{\alpha_{f_5,i}}+$$

$$\left.+2^{2\alpha_{l,1}+2\alpha_{K,1}+1}\cdot\prod_{i=2}^{\omega(K)}p_i^{2\alpha_{l,i}+2\alpha_{K,i}}\cdot 5^{\alpha_{f_5,\omega(K)+1}+1}\cdot\prod_{i=\omega(K)+2}^{\omega(f_5)}p_{i,f_5}^{\alpha_{f_5,i}}\right\}\in\mathbb{N}^+\quad(152)$$

A sufficient condition for the expression in (152) to be a natural positive number is:

$$\alpha_{l,1}-2\alpha_{f_5,1}+\min\left\{\alpha_{f_3,1}+\alpha_{f_5,1},\alpha_{l,1}+\alpha_{K,1}+\alpha_{f_4,1}+\alpha_{f_5,1}+1,2\alpha_{l,1}+2\alpha_{K,1}+1\right\}\geq 0$$

$$\Leftrightarrow\alpha_{l,1}\geq 2\alpha_{f_5,1}-\min\left\{\alpha_{f_3,1}+\alpha_{f_5,1},\alpha_{l,1}+\alpha_{K,1}+\alpha_{f_4,1}+\alpha_{f_5,1}+1,2\alpha_{l,1}+2\alpha_{K,1}+1\right\}\ (153)$$

$$\alpha_{l,i}-2\alpha_{f_5,i}+\min\left\{\alpha_{f_3,i}+\alpha_{f_5,i},\alpha_{l,i}+\alpha_{K,i}+\alpha_{f_4,i}+\alpha_{f_5,i},2\alpha_{l,i}+2\alpha_{K,i}\right\}\geq 0$$

$$\Leftrightarrow \alpha_{l,i}\geq 2\alpha_{f_5,i}-\min\left\{\alpha_{f_3,i}+\alpha_{f_5,i},\alpha_{l,i}+\alpha_{K,i}+\alpha_{f_4,i}+\alpha_{f_5,i},2\alpha_{l,i}+2\alpha_{K,i}\right\} \quad (154a)$$

for $i=2,3,\ldots,\omega(K)$, dacă $3\nmid K$.

If $3\mid K$, that is $p_2=3$, then, for $i=2$, the relation (154a) becomes

$$\alpha_{l,2}\geq 2\alpha_{f_5,2}-\min\left\{\alpha_{f_3,2}+\alpha_{f_5,2}+1,\alpha_{l,2}+\alpha_{K,2}+\alpha_{f_4,2}+\alpha_{f_5,2}+1,2\alpha_{l,2}+2\alpha_{K,2}\right\} \quad (154b)$$

II) If $5\mid K$, considering (139), we have

$$\frac{l\cdot\left(5\cdot 3\cdot f_3\cdot f_5+5\cdot 2\cdot 3\cdot f_4\cdot f_5\cdot l\cdot K+2\cdot l^2\cdot K^2\right)}{5^2\cdot f_5^2}=2^{\alpha_{l,1}+\alpha_{f_3,1}-\alpha_{f_5,1}}\cdot 3\cdot 5^{\alpha_{l,2}+\alpha_{f_3,2}-\alpha_{f_5,2}-1}\cdot$$

$$\cdot\prod_{i=3}^{\omega(K)}p_i^{\alpha_{l,i}+\alpha_{f_3,i}-\alpha_{f_5,i}}\cdot\prod_{i=\omega(K)+1}^{\omega(f_3)}p_{i,f_3}^{\alpha_{f_3,i}}\cdot\prod_{i=\omega(K)+1}^{\omega(f_5)}p_{i,f_5}^{\alpha_{l,i}-\alpha_{f_5,i}}+2^{2\alpha_{l,1}+\alpha_{K,1}+\alpha_{f_4,1}-\alpha_{f_5,1}+1}\cdot 3\cdot 5^{2\alpha_{l,2}+\alpha_{K,2}+\alpha_{f_4,2}-\alpha_{f_5,2}-1}\cdot$$

$$\cdot\prod_{i=3}^{\omega(K)}p_i^{2\alpha_{l,i}+\alpha_{K,i}+\alpha_{f_4,i}-\alpha_{f_5,i}}\cdot\prod_{i=\omega(K)+1}^{\omega(f_4)}p_{i,f_4}^{\alpha_{f_4,i}}\cdot\prod_{i=\omega(K)+1}^{\omega(f_5)}p_{i,f_5}^{2\alpha_{l,i}-\alpha_{f_5,i}}+2^{3\alpha_{l,1}+2\alpha_{K,1}-2\alpha_{f_5,1}+1}\cdot 5^{3\alpha_{l,2}+2\alpha_{K,2}-2\alpha_{f_5,2}-2}\cdot$$

$$\cdot\prod_{i=3}^{\omega(K)}p_i^{3\alpha_{l,i}+2\alpha_{K,i}-2\alpha_{f_5,i}}\cdot\prod_{i=\omega(K)+1}^{\omega(f_5)}p_{i,f_5}^{3\alpha_{l,i}-2\alpha_{5,i}}=2^{\alpha_{l,1}-2\alpha_{f_5,1}}\cdot 5^{\alpha_{l,2}-2\alpha_{f_5,2}-2}\cdot\prod_{i=3}^{\omega(K)}p_i^{\alpha_{l,i}-2\alpha_{f_5,i}}\cdot$$

$$\left\{2^{\alpha_{f_3,1}+\alpha_{f_5,1}}\cdot 3\cdot 5^{\alpha_{f_3,2}+\alpha_{f_5,2}+1}\cdot\prod_{i=3}^{\omega(K)}p_i^{\alpha_{f_3,i}+\alpha_{f_5,i}}\cdot\prod_{i=\omega(K)+1}^{\omega(f_3)}p_{i,f_3}^{\alpha_{f_3,i}}+2^{\alpha_{l,1}+\alpha_{K,1}+\alpha_{f_4,1}+\alpha_{f_5,1}+1}\cdot 3\cdot\right.$$

$$\cdot 5^{\alpha_{l,2}+\alpha_{K,2}+\alpha_{f_4,2}+\alpha_{f_5,2}+1}\cdot\prod_{i=3}^{\omega(K)}p_i^{\alpha_{l,i}+\alpha_{K,i}+\alpha_{f_4,i}+\alpha_{f_5,i}}\cdot\prod_{i=\omega(K)+1}^{\omega(f_4)}p_{i,f_4}^{\alpha_{f_4,i}}\cdot\prod_{i=\omega(K)+1}^{\omega(f_5)}p_{i,f_5}^{\alpha_{f_5,i}}+$$

$$\left.+2^{2\alpha_{l,1}+2\alpha_{K,1}+1}\cdot 5^{2\alpha_{l,2}+2\alpha_{K,2}}\cdot\prod_{i=3}^{\omega(K)}p_i^{2\alpha_{l,i}+2\alpha_{K,i}}\cdot\prod_{i=\omega(K)+1}^{\omega(f_5)}p_{i,f_5}^{\alpha_{f_5,i}}\right\}\in\mathbb{N}^+ \quad (155)$$

A sufficient condition for the expression in (155) to be a natural positive number is:

$$\Leftrightarrow \alpha_{l,1}\geq 2\alpha_{f_5,1}-\min\left\{\alpha_{f_3,1}+\alpha_{f_5,1},\alpha_{l,1}+\alpha_{K,1}+\alpha_{f_4,1}+\alpha_{f_5,1}+1,2\alpha_{l,1}+2\alpha_{K,1}+1\right\} \quad (156)$$

$$\alpha_{l,2}\geq 2\alpha_{f_5,2}+2-\min\left\{\alpha_{f_3,2}+\alpha_{f_5,2}+1,\alpha_{l,2}+\alpha_{K,2}+\alpha_{f_4,2}+\alpha_{f_5,2}+1,2\alpha_{l,2}+2\alpha_{K,2}\right\}$$

(157)

$$\Leftrightarrow \alpha_{l,i}\geq 2\alpha_{f_5,i}-\min\left\{\alpha_{f_3,i}+\alpha_{f_5,i},\alpha_{l,i}+\alpha_{K,i}+\alpha_{f_4,i}+\alpha_{f_5,i},2\alpha_{l,i}+2\alpha_{K,i}\right\} \quad (158a)$$

for $i=3,\ldots,\omega(K)$, dacă $3\nmid K$.

If $3\mid K$, that is $p_2=3$, then, for $i=3$, the relation (158a) becomes

$$\alpha_{l,3}\geq 2\alpha_{f_5,3}-\min\left\{\alpha_{f_3,3}+\alpha_{f_5,3}+1,\alpha_{l,3}+\alpha_{K,3}+\alpha_{f_4,3}+\alpha_{f_5,3}+1,2\alpha_{l,3}+2\alpha_{K,3}\right\} \quad (158b)$$

From (31), we have:

I) If $5\nmid K$, considering conditions (135)-(136), we have:

$$\frac{l\cdot\left(5^2\cdot 2\cdot f_2\cdot f_5^2+5\cdot 3\cdot f_3\cdot f_5\cdot l\cdot K+4\cdot f_4\cdot l^2\cdot K^2+l^3\cdot K^3\right)}{5^3\cdot f_5^3}=2^{\alpha_{l,1}+\alpha_{f_2,1}-\alpha_{f_5,1}+1}\cdot$$

$$\cdot\prod_{i=2}^{\omega(K)}p_i^{\alpha_{l,i}+\alpha_{f_2,i}-\alpha_{f_5,i}}\cdot\prod_{i=\omega(K)+1}^{\omega(f_2)}p_{i,f_2}^{\alpha_{f_2,i}}\cdot 5^{\alpha_{l,\omega(K)+1}-\alpha_{f_5,\omega(K)+1}-1}\cdot\prod_{i=\omega(K)+2}^{\omega(f_5)}p_{i,f_5}^{\alpha_{l,i}-\alpha_{f_5,i}}+2^{2\alpha_{l,1}+\alpha_{K,1}+\alpha_{f_3,1}-2\alpha_{f_5,1}}\cdot 3\cdot$$

$$\cdot\prod_{i=2}^{\omega(K)}p_i^{2\alpha_{l,i}+\alpha_{K,i}+\alpha_{f_3,i}-2\alpha_{f_5,i}}\cdot\prod_{i=\omega(K)+1}^{\omega(f_3)}p_{i,f_3}^{\alpha_{f_3,i}}\cdot 5^{2\alpha_{l,\omega(K)+1}-2\alpha_{f_5,\omega(K)+1}-2}\cdot\prod_{i=\omega(K)+2}^{\omega(f_5)}p_{i,f_5}^{2\alpha_{l,i}-2\alpha_{f_5,i}}+$$

$$+2^{3\alpha_{l,1}+2\alpha_{K,1}+\alpha_{f_4,1}-3\alpha_{f_5,1}+2}\cdot\prod_{i=2}^{\omega(K)}p_i^{3\alpha_{l,i}+2\alpha_{K,i}+\alpha_{f_4,i}-3\alpha_{f_5,i}}\cdot\prod_{i=\omega(K)+1}^{\omega(f_4)}p_{i,f_4}^{\alpha_{f_4,i}}\cdot 5^{3\alpha_{l,\omega(K)+1}-3\alpha_{f_5,\omega(K)+1}-3}\cdot$$

$$\cdot\prod_{i=\omega(K)+2}^{\omega(f_5)}p_{i,f_5}^{3\alpha_{l,i}-3\alpha_{f_5,i}}+2^{4\alpha_{l,1}+3\alpha_{K,1}-3\alpha_{f_5,1}}\cdot\prod_{i=2}^{\omega(K)}p_i^{4\alpha_{l,i}+3\alpha_{K,i}-3\alpha_{f_5,i}}\cdot 5^{4\alpha_{l,\omega(K)+1}-3\alpha_{f_5,\omega(K)+1}-3}\cdot$$

$$\cdot\prod_{i=\omega(K)+2}^{\omega(f_5)}p_{i,f_5}^{4\alpha_{l,i}-3\alpha_{f_5,i}}=2^{\alpha_{l,1}-3\alpha_{f_5,1}}\cdot\prod_{i=2}^{\omega(K)}p_i^{\alpha_{l,i}-3\alpha_{f_5,i}}\cdot\left\{2^{\alpha_{f_2,1}+2\alpha_{f_5,1}+1}\cdot\prod_{i=2}^{\omega(K)}p_i^{\alpha_{f_2,i}+2\alpha_{f_5,i}}\cdot\prod_{i=\omega(K)+1}^{\omega(f_2)}p_{i,f_2}^{\alpha_{f_2,i}}+\right.$$

$$+2^{\alpha_{l,1}+\alpha_{K,1}+\alpha_{f_3,1}+\alpha_{f_5,1}}\cdot 3\cdot\prod_{i=2}^{\omega(K)}p_i^{\alpha_{l,i}+\alpha_{K,i}+\alpha_{f_3,i}+\alpha_{f_5,i}}\cdot\prod_{i=\omega(K)+1}^{\omega(f_3)}p_{i,f_3}^{\alpha_{f_3,i}}+2^{2\alpha_{l,1}+2\alpha_{K,1}+\alpha_{f_4,1}+2}\cdot$$

$$\left.\cdot\prod_{i=2}^{\omega(K)}p_i^{2\alpha_{l,i}+2\alpha_{K,i}+\alpha_{f_4,i}}\cdot\prod_{i=\omega(K)+1}^{\omega(f_4)}p_{i,f_4}^{\alpha_{f_4,i}}+2^{3\alpha_{l,1}+3\alpha_{K,1}}\cdot\prod_{i=2}^{\omega(K)}p_i^{3\alpha_{l,i}+3\alpha_{K,i}}\cdot 5^{\alpha_{f_5,\omega(K)+1}+1}\cdot\prod_{i=\omega(K)+2}^{\omega(f_5)}p_{i,f_5}^{\alpha_{f_5,i}}\right\}\in\mathbb{N}^+$$

(159)

A sufficient condition for the expression in (159) to be a natural positive number is:

$$\alpha_{l,1}-3\alpha_{f_5,1}+$$
$$+\min\left\{\alpha_{f_2,1}+2\alpha_{f_5,1}+1,\alpha_{l,1}+\alpha_{K,1}+\alpha_{f_3,1}+\alpha_{f_5,1},2\alpha_{l,1}+2\alpha_{K,1}+\alpha_{f_4,1}+2,3\alpha_{l,1}+3\alpha_{K,1}\right\}\geq 0$$
$$\Leftrightarrow \alpha_{l,1}\geq 3\alpha_{f_5,1}-$$
$$-\min\left\{\alpha_{f_2,1}+2\alpha_{f_5,1}+1,\alpha_{l,1}+\alpha_{K,1}+\alpha_{f_3,1}+\alpha_{f_5,1},2\alpha_{l,1}+2\alpha_{K,1}+\alpha_{f_4,1}+2,3\alpha_{l,1}+3\alpha_{K,1}\right\}$$

(160)

$$\alpha_{l,i}-3\alpha_{f_5,i}+$$
$$+\min\left\{\alpha_{f_2,i}+2\alpha_{f_5,i},\alpha_{l,i}+\alpha_{K,i}+\alpha_{f_3,i}+\alpha_{f_5,i},2\alpha_{l,i}+2\alpha_{K,i}+\alpha_{f_4,i},3\alpha_{l,i}+3\alpha_{K,i}\right\}\geq 0$$
$$\Leftrightarrow \alpha_{l,i}\geq 3\alpha_{f_5,i}-$$
$$-\min\left\{\alpha_{f_2,i}+2\alpha_{f_5,i},\alpha_{l,i}+\alpha_{K,i}+\alpha_{f_3,i}+\alpha_{f_5,i},2\alpha_{l,i}+2\alpha_{K,i}+\alpha_{f_4,i},3\alpha_{l,i}+3\alpha_{K,i}\right\} \quad (161a)$$

for $i=2,3,\ldots,\omega(K)$, dacă $3\nmid K$.

If $3\mid K$, that is $p_2=3$, then, for $i=2$, the relation (161a) becomes

$$\alpha_{l,2}\geq 3\alpha_{f_5,2}-$$
$$-\min\left\{\alpha_{f_2,2}+2\alpha_{f_5,2},\alpha_{l,2}+\alpha_{K,2}+\alpha_{f_3,2}+\alpha_{f_5,2}+1,2\alpha_{l,2}+2\alpha_{K,2}+\alpha_{f_4,2},3\alpha_{l,2}+3\alpha_{K,2}\right\}$$

(161b)

II) If $5\mid K$, considering (139), we have

$$\frac{l\cdot\left(5^2\cdot 2\cdot f_2\cdot f_5^2+5\cdot 3\cdot f_3\cdot f_5\cdot l\cdot K+4\cdot f_4\cdot l^2\cdot K^2+l^3\cdot K^3\right)}{5^3\cdot f_5^3}=2^{\alpha_{l,1}+\alpha_{f_2,1}-\alpha_{f_5,1}+1}\cdot$$

$$\cdot 5^{\alpha_{l,2}+\alpha_{f_2,2}-\alpha_{f_5,2}-1}\cdot\prod_{i=3}^{\omega(K)}p_i^{\alpha_{l,i}+\alpha_{f_2,i}-\alpha_{f_5,i}}\cdot\prod_{i=\omega(K)+1}^{\omega(f_2)}p_{i,f_2}^{\alpha_{f_2,i}}\cdot\prod_{i=\omega(K)+1}^{\omega(f_5)}p_{i,f_5}^{\alpha_{l,i}-\alpha_{f_5,i}}+2^{2\alpha_{l,1}+\alpha_{K,1}+\alpha_{f_3,1}-2\alpha_{f_5,1}}\cdot 3\cdot$$

$$\cdot 5^{2\alpha_{l,2}+\alpha_{K,2}+\alpha_{f_3,2}-2\alpha_{f_5,2}-2}\cdot\prod_{i=3}^{\omega(K)}p_i^{2\alpha_{l,i}+\alpha_{K,i}+\alpha_{f_3,i}-2\alpha_{f_5,i}}\cdot\prod_{i=\omega(K)+1}^{\omega(f_3)}p_{i,f_3}^{\alpha_{f_3,i}}\cdot\prod_{i=\omega(K)+1}^{\omega(f_5)}p_{i,f_5}^{2\alpha_{l,i}-2\alpha_{f_5,i}}+$$

$$+2^{3\alpha_{l,1}+2\alpha_{K,1}+\alpha_{f_4,1}-3\alpha_{f_5,1}+2}\cdot 5^{3\alpha_{l,2}+2\alpha_{K,2}+\alpha_{f_4,2}-3\alpha_{f_5,2}-3}\cdot\prod_{i=3}^{\omega(K)}p_i^{3\alpha_{l,i}+2\alpha_{K,i}+\alpha_{f_4,i}-3\alpha_{f_5,i}}\cdot\prod_{i=\omega(K)+1}^{\omega(f_4)}p_{i,f_4}^{\alpha_{f_4,i}}\cdot$$

$$\cdot \prod_{i=\omega(K)+1}^{\omega(f_5)} p_{i,f_5}^{3\alpha_{l,i}-3\alpha_{f_5,i}} + 2^{4\alpha_{l,1}+3\alpha_{K,1}-3\alpha_{f_5,1}} \cdot 5^{4\alpha_{l,2}+3\alpha_{K,2}-3\alpha_{f_5,2}-3} \cdot \prod_{i=3}^{\omega(K)} p_i^{4\alpha_{l,i}+3\alpha_{K,i}-3\alpha_{f_5,i}} \cdot$$

$$\cdot \prod_{i=\omega(K)+1}^{\omega(f_5)} p_{i,f_5}^{4\alpha_{l,i}-3\alpha_{f_5,i}} = 2^{\alpha_{l,1}-3\alpha_{f_5,1}} \cdot 5^{\alpha_{l,2}-3\alpha_{f_5,2}-3} \cdot \prod_{i=3}^{\omega(K)} p_i^{\alpha_{l,i}-3\alpha_{f_5,i}} \cdot \Big\{ 2^{\alpha_{f_2,1}+2\alpha_{f_5,1}+1} \cdot 5^{\alpha_{f_2,2}+2\alpha_{f_5,2}+2} \cdot$$

$$\cdot \prod_{i=3}^{\omega(K)} p_i^{\alpha_{f_2,i}+2\alpha_{f_5,i}} \cdot \prod_{i=\omega(K)+1}^{\omega(f_2)} p_{i,f_2}^{\alpha_{f_2,i}} + 2^{\alpha_{l,1}+\alpha_{K,1}+\alpha_{f_3,1}+\alpha_{f_5,1}} \cdot 3 \cdot 5^{\alpha_{l,2}+\alpha_{K,2}+\alpha_{f_3,2}+\alpha_{f_5,2}+1} \cdot$$

$$\cdot \prod_{i=3}^{\omega(K)} p_i^{\alpha_{l,i}+\alpha_{K,i}+\alpha_{f_3,i}+\alpha_{f_5,i}} \cdot \prod_{i=\omega(K)+1}^{\omega(f_3)} p_{i,f_3}^{\alpha_{f_3,i}} + 2^{2\alpha_{l,1}+2\alpha_{K,1}+\alpha_{f_4,1}+2} \cdot 5^{2\alpha_{l,2}+2\alpha_{K,2}+\alpha_{f_4,2}} \cdot \prod_{i=3}^{\omega(K)} p_i^{2\alpha_{l,i}+2\alpha_{K,i}+\alpha_{f_4,i}} \cdot$$

$$\cdot \prod_{i=\omega(K)+1}^{\omega(f_4)} p_{i,f_4}^{\alpha_{f_4,i}} + 2^{3\alpha_{l,1}+3\alpha_{K,1}} \cdot 5^{3\alpha_{l,2}+3\alpha_{K,2}} \cdot \prod_{i=3}^{\omega(K)} p_i^{3\alpha_{l,i}+3\alpha_{K,i}} \cdot \prod_{i=\omega(K)+1}^{\omega(f_5)} p_{i,f_5}^{\alpha_{f_5,i}} \Big\} \in \mathbb{N}^+ \quad (162)$$

A sufficient condition for the expression in (162) to be a natural positive number is:

$$\Leftrightarrow \alpha_{l,1} \geq 3\alpha_{f_5,1} -$$
$$-\min\left\{\alpha_{f_2,1}+2\alpha_{f_5,1}+1, \alpha_{l,1}+\alpha_{K,1}+\alpha_{f_3,1}+\alpha_{f_5,1}, 2\alpha_{l,1}+2\alpha_{K,1}+\alpha_{f_4,1}+2, 3\alpha_{l,1}+3\alpha_{K,1}\right\}$$

(163)

$$\alpha_{l,2} \geq 3\alpha_{f_5,2} + 3 -$$
$$-\min\left\{\alpha_{f_2,2}+2\alpha_{f_5,2}+2, \alpha_{l,2}+\alpha_{K,2}+\alpha_{f_3,2}+\alpha_{f_5,2}+1, 2\alpha_{l,2}+2\alpha_{K,2}+\alpha_{f_4,2}, 3\alpha_{l,2}+3\alpha_{K,2}\right\}$$

(164)

$$\alpha_{l,i} - 3\alpha_{f_5,i} +$$
$$+\min\left\{\alpha_{f_2,i}+2\alpha_{f_5,i}, \alpha_{l,i}+\alpha_{K,i}+\alpha_{f_3,i}+\alpha_{f_5,i}, 2\alpha_{l,i}+2\alpha_{K,i}+\alpha_{f_4,i}, 3\alpha_{l,i}+3\alpha_{K,i}\right\} \geq 0$$
$$\Leftrightarrow \alpha_{l,i} \geq 3\alpha_{f_5,i} -$$
$$-\min\left\{\alpha_{f_2,i}+2\alpha_{f_5,i}, \alpha_{l,i}+\alpha_{K,i}+\alpha_{f_3,i}+\alpha_{f_5,i}, 2\alpha_{l,i}+2\alpha_{K,i}+\alpha_{f_4,i}, 3\alpha_{l,i}+3\alpha_{K,i}\right\} \quad (165a)$$

for $i = 3,\ldots,\omega(K)$ $i = 3,\ldots,\omega(K)$, if $3 \nmid K$.

If $3 \mid K$, that is $p_3 = 3$, then, for $i = 3$, the relation (165a) becomes

$$\alpha_{l,3} \geq 3\alpha_{f_5,3} -$$
$$-\min\left\{\alpha_{f_2,3}+2\alpha_{f_5,3}, \alpha_{l,3}+\alpha_{K,3}+\alpha_{f_3,3}+\alpha_{f_5,3}+1, 2\alpha_{l,3}+2\alpha_{K,3}+\alpha_{f_4,3}, 3\alpha_{l,3}+3\alpha_{K,3}\right\}$$

(165b)

Concluding,

I) If $5 \nmid K$, gathering the conditions (134)-(136), (141), (146)-(147), (153)-(154a,b) and (160)- (161a,b), we have

$$\alpha_{f_5,1}-$$

$$-\min\left\{\begin{array}{l}\alpha_{K,1},\\ \dfrac{\alpha_{K,1}-2\alpha_{f_5,1}}{2}+\\ +\dfrac{\min\left\{\alpha_{f_2,1}+2\alpha_{f_5,1},\alpha_{l,1}+\alpha_{K,1}+\alpha_{f_3,1}+\alpha_{f_5,1},2\alpha_{l,1}+2\alpha_{K,1}+\alpha_{f_4,1},3\alpha_{l,1}+3\alpha_{K,1}\right\}}{2},\\ 1+\min\left\{\alpha_{f_4,1}+1,\alpha_{l,1}+\alpha_{K,1}\right\},\\ -\alpha_{f_5,1}+\min\left\{\alpha_{f_3,1}+\alpha_{f_5,1},\alpha_{l,1}+\alpha_{K,1}+\alpha_{f_5,1}+1,2\alpha_{l,1}+2\alpha_{K,1}+1\right\},\\ -2\alpha_{f_5,1}+\min\left\{\begin{array}{l}\alpha_{f_2,1}+2\alpha_{f_5,1}+1,\alpha_{l,1}+\alpha_{K,1}+\alpha_{f_3,1}+\alpha_{f_5,1},\\ 2\alpha_{l,1}+2\alpha_{K,1}+\alpha_{f_4,1}+2,3\alpha_{l,1}+3\alpha_{K,1}\end{array}\right\}\end{array}\right\}\leq$$

$$\leq\alpha_{l,1}\leq\alpha_{f_5,1}$$

(166)

$$\alpha_{f_5,i}-$$

$$-\min\left\{\begin{array}{l}\alpha_{K,i},\\ \dfrac{\alpha_{K,i}-2\alpha_{f_5,i}}{2}+\\ +\dfrac{\min\left\{\alpha_{f_2,i}+2\alpha_{f_5,i},\alpha_{l,i}+\alpha_{K,i}+\alpha_{f_3,i}+\alpha_{f_5,i},2\alpha_{l,i}+2\alpha_{K,i}+\alpha_{f_4,i},3\alpha_{l,i}+3\alpha_{K,i}\right\}}{2},\\ \min\left\{\alpha_{f_4,i},\alpha_{l,i}+\alpha_{K,i}\right\},\\ -\alpha_{f_5,i}+\min\left\{\alpha_{f_3,i}+\alpha_{f_5,i},\alpha_{l,i}+\alpha_{K,i}+\alpha_{f_5,i},2\alpha_{l,i}+2\alpha_{K,i}\right\},\\ -2\alpha_{f_5,i}+\min\left\{\begin{array}{l}\alpha_{f_2,i}+2\alpha_{f_5,i},\alpha_{l,i}+\alpha_{K,i}+\alpha_{f_3,i}+\alpha_{f_5,i},\\ 2\alpha_{l,i}+2\alpha_{K,i}+\alpha_{f_4,i},3\alpha_{l,i}+3\alpha_{K,i}\end{array}\right\}\end{array}\right\}\leq$$

$$\leq\alpha_{l,i}\leq\alpha_{f_5,i}$$

(167a)

for $i=2,3,\ldots,\omega(K)$, if $3\nmid K$.

If $3\mid K$, that is $p_2=3$, then, for $i=2$, the relation (167a) becomes

$$\alpha_{f_5,2} - \min\left\{\begin{array}{l} \alpha_{K,2}, \\ \frac{\alpha_{K,2}-2\alpha_{f_5,2}}{2} + \\ +\frac{\min\left\{\alpha_{f_2,2}+2\alpha_{f_5,2},\alpha_{l,2}+\alpha_{K,2}+\alpha_{f_3,2}+\alpha_{f_5,2},2\alpha_{l,2}+2\alpha_{K,2}+\alpha_{f_4,2},3\alpha_{l,2}+3\alpha_{K,2}\right\}}{2}, \\ \min\left\{\alpha_{f_4,2},\alpha_{l,2}+\alpha_{K,2}\right\}, \\ -\alpha_{f_5,2}+\min\left\{\alpha_{f_3,2}+\alpha_{f_5,2}+1,\alpha_{l,2}+\alpha_{K,2}+\alpha_{f_5,2}+1,2\alpha_{l,2}+2\alpha_{K,2}\right\}, \\ -2\alpha_{f_5,2} + \\ +\min\left\{\alpha_{f_2,2}+2\alpha_{f_5,2},\alpha_{l,2}+\alpha_{K,2}+\alpha_{f_3,2}+\alpha_{f_5,2}+1,2\alpha_{l,2}+2\alpha_{K,2}+\alpha_{f_4,2},3\alpha_{l,2}+3\alpha_{K,2}\right\} \end{array}\right\} \le$$

$$\le \alpha_{l,2} \le \alpha_{f_5,2}$$

(167b)

$$\alpha_{l,\omega(K)+1} = \alpha_{f_5,\omega(K)+1} + 1, \qquad (168)$$

and

$$\alpha_{l,i} = \alpha_{f_5,i}, \qquad (169)$$

for $i = \omega(K)+2,\ldots,\omega(f_5)$.

II) If $5|K$, gathering the conditions (137)-(139), (143)-(144), (149)-(151), (156)-(158) and (163)- (165), we have

$$\alpha_{f_5,1} - \min\left\{\begin{array}{l} \alpha_{K,1}, \\ \frac{\alpha_{K,1}-2\alpha_{f_5,1}}{2} + \\ +\frac{\min\left\{\alpha_{f_2,1}+2\alpha_{f_5,1},\alpha_{l,1}+\alpha_{K,1}+\alpha_{f_3,1}+\alpha_{f_5,1},2\alpha_{l,1}+2\alpha_{K,1}+\alpha_{f_4,1},3\alpha_{l,1}+3\alpha_{K,1}\right\}}{2}, \\ 1+\min\left\{\alpha_{f_4,1}+1,\alpha_{l,1}+\alpha_{K,1}\right\}, \\ -\alpha_{f_5,1}+\min\left\{\alpha_{f_3,1}+\alpha_{f_5,1},\alpha_{l,1}+\alpha_{K,1}+\alpha_{f_5,1}+1,2\alpha_{l,1}+2\alpha_{K,1}+1\right\}, \\ -2\alpha_{f_5,1}+\min\left\{\begin{array}{l}\alpha_{f_2,1}+2\alpha_{f_5,1}+1,\alpha_{l,1}+\alpha_{K,1}+\alpha_{f_3,1}+\alpha_{f_5,1}, \\ 2\alpha_{l,1}+2\alpha_{K,1}+\alpha_{f_4,1}+2,3\alpha_{l,1}+3\alpha_{K,1}\end{array}\right\} \end{array}\right\} \le$$

$$\le \alpha_{l,1} \le \alpha_{f_5,1}$$

(170)

$$\alpha_{f_5,2}+1-$$

$$-\min\left\{\begin{array}{l}\alpha_{K,2},\\ \dfrac{\alpha_{K,2}-2\alpha_{f_5,2}-3}{2}+\\ +\dfrac{1}{2}\cdot\min\left\{\begin{array}{l}\alpha_{f_2,2}+2\alpha_{f_5,2}+3,\alpha_{l,2}+\alpha_{K,2}+\alpha_{f_3,2}+\alpha_{f_5,2}+2,\\ 2\alpha_{l,2}+2\alpha_{K,2}+\alpha_{f_4,2}+1,3\alpha_{l,2}+3\alpha_{K,2}\end{array}\right\},\\ \min\left\{\alpha_{f_4,2},\alpha_{l,2}+\alpha_{K,2}\right\},\\ -\alpha_{f_5,2}-1+\min\left\{\alpha_{f_3,2}+\alpha_{f_5,2}+1,\alpha_{l,2}+\alpha_{K,2}+\alpha_{f_4,2}+\alpha_{f_5,2}+1,2\alpha_{l,2}+2\alpha_{K,2}\right\},\\ -2\alpha_{f_5,2}-2+\\ +\min\left\{\begin{array}{l}\alpha_{f_2,2}+2\alpha_{f_5,2}+2,\alpha_{l,2}+\alpha_{K,2}+\alpha_{f_3,2}+\alpha_{f_5,2}+1,\\ 2\alpha_{l,2}+2\alpha_{K,2}+\alpha_{f_4,2},3\alpha_{l,2}+3\alpha_{K,2}\end{array}\right\}\end{array}\right\}\le$$

$$\le\alpha_{l,2}\le\alpha_{f_5,2}+1$$

(171)

$$\alpha_{f_5,i}-$$

$$-\min\left\{\begin{array}{l}\alpha_{K,i},\\ \dfrac{\alpha_{K,i}-2\alpha_{f_5,i}}{2}+\\ +\dfrac{\min\left\{\alpha_{f_2,i}+2\alpha_{f_5,i},\alpha_{l,i}+\alpha_{K,i}+\alpha_{f_3,i}+\alpha_{f_5,i},2\alpha_{l,i}+2\alpha_{K,i}+\alpha_{f_4,i},3\alpha_{l,i}+3\alpha_{K,i}\right\}}{2},\\ \min\left\{\alpha_{f_4,i},\alpha_{l,i}+\alpha_{K,i}\right\},\\ -\alpha_{f_5,i}+\min\left\{\alpha_{f_3,i}+\alpha_{f_5,i},\alpha_{l,i}+\alpha_{K,i}+\alpha_{f_5,i},2\alpha_{l,i}+2\alpha_{K,i}\right\},\\ -2\alpha_{f_5,i}+\min\left\{\begin{array}{l}\alpha_{f_2,i}+2\alpha_{f_5,i},\alpha_{l,i}+\alpha_{K,i}+\alpha_{f_3,i}+\alpha_{f_5,i},\\ 2\alpha_{l,i}+2\alpha_{K,i}+\alpha_{f_4,i},3\alpha_{l,i}+3\alpha_{K,i}\end{array}\right\}\end{array}\right\}\le$$

$$\le\alpha_{l,i}\le\alpha_{f_5,i}$$

(172a)

for $i=3,\ldots,\omega(K)$, if $3\nmid K$.

If $3\mid K$, that is $p_3=3$, then, for $i=3$, the relation (172a) becomes

$$\alpha_{f_5,3} - $$

$$-\min\left\{\begin{array}{l}\alpha_{K,3},\\ \dfrac{\alpha_{K,3}-2\alpha_{f_5,3}}{2}+\\ +\dfrac{\min\left\{\alpha_{f_2,3}+2\alpha_{f_5,3},\alpha_{l,3}+\alpha_{K,3}+\alpha_{f_3,3}+\alpha_{f_5,3},2\alpha_{l,3}+2\alpha_{K,3}+\alpha_{f_4,3},3\alpha_{l,3}+3\alpha_{K,3}\right\}}{2},\\ \min\left\{\alpha_{f_4,3},\alpha_{l,3}+\alpha_{K,3}\right\},\\ -\alpha_{f_5,3}+\min\left\{\alpha_{f_3,3}+\alpha_{f_5,3}+1,\alpha_{l,3}+\alpha_{K,3}+\alpha_{f_5,3}+1,2\alpha_{l,3}+2\alpha_{K,3}\right\},\\ -2\alpha_{f_5,3}+\\ +\min\left\{\begin{array}{l}\alpha_{f_2,3}+2\alpha_{f_5,3},\alpha_{l,3}+\alpha_{K,3}+\alpha_{f_3,3}+\alpha_{f_5,3}+1,\\ 2\alpha_{l,3}+2\alpha_{K,3}+\alpha_{f_4,3},3\alpha_{l,3}+3\alpha_{K,3}\end{array}\right\}\end{array}\right\}\le$$

$$\le \alpha_{l,3} \le \alpha_{f_5,3}$$

(172b)

and

$$\alpha_{l,i} = \alpha_{f_5,i}, \qquad (173)$$

for $i = \omega(K)+1,\ldots,\omega(f_5)$.

We mention that if the left hand side of the double inequalities (166)-(167a,b) and (170)-(172a,b) is negative, it is considered to be 0.

Obviously, for $\alpha_{l,i} = \alpha_{f_5,i}$, with $i = 1,2,3,\ldots,\omega(K)$, when $5 \nmid K$ and for $\alpha_{l,i} = \alpha_{f_5,i}$, with $i = 1,3,\ldots,\omega(K)$ and $\alpha_{l,2} = \alpha_{f_5,2}+1$, when $5 \mid K$, it results that $Q = K$.

To demonstate that there are also valid values $Q < K$, it is sufficient to prove that $\alpha_{l,i} = \alpha_{f_5,i} - 1$ verifies the first inequality in (167a) or (172a). Because for $i = 2,3,\ldots,\omega(K)$, we have $\alpha_{f_2,i} \ge 1$, $\alpha_{f_3,i} \ge 1$, $\alpha_{f_4,i} \ge 1$, $\alpha_{f_5,i} \ge 1$, $\alpha_{K,i} \ge 1$, then, replacing-$\alpha_{l,i} = \alpha_{f_4,i} - 1$ in the left hand side of (167a) or (172a) leads to

$$\alpha_{f_5,i} - $$

$$-\min\left\{\begin{array}{l}\alpha_{K,i},\\ \dfrac{\alpha_{K,i}-2\alpha_{f_5,i}}{2}+\\ +\dfrac{\min\left\{\alpha_{f_2,i}+2\alpha_{f_5,i},\alpha_{K,i}+\alpha_{f_3,i}+2\alpha_{f_5,i}-1,2\alpha_{K,i}+\alpha_{f_4,i}+2\alpha_{f_5,i}-2,3\alpha_{K,i}+3\alpha_{f_5,i}-3\right\}}{2},\\ \min\left\{\alpha_{f_4,i},\alpha_{K,i}+\alpha_{f_5,i}-1\right\},\\ -\alpha_{f_5,i}+\min\left\{\alpha_{f_3,i}+\alpha_{f_5,i},\alpha_{K,i}+2\alpha_{f_5,i}-1,2\alpha_{K,i}+2\alpha_{f_5,i}-2\right\},\\ -2\alpha_{f_5,i}+\\ +\min\left\{\alpha_{f_2,i}+2\alpha_{f_5,i},\alpha_{K,i}+\alpha_{f_3,i}+2\alpha_{f_5,i}-1,2\alpha_{K,i}+\alpha_{f_4,i}+2\alpha_{f_5,i}-2,3\alpha_{K,i}+3\alpha_{f_5,i}-3\right\}\end{array}\right\}\leq$$

$$\leq \alpha_{f_5,i}-1$$

(174)

Therefore, for any 5-PP satisfying the conditions imposed by theorem 1 from [2] there exists an equivalent ARP with $Q < K$.

<u>Example 9</u>

Let $K = 1696 = 2^5 \cdot 53^1$ and the 5-PP coefficients $f_1 = 55 = 5^1 \cdot 11^1$, $f_2 = 954 = 2^1 \cdot 53^1 \cdot 3^2$, $f_3 = 1272 = 2^3 \cdot 53^1 \cdot 3^1$, $f_4 = 1484 = 2^2 \cdot 53^1 \cdot 7^1$, $f_5 = 1060 = 2^2 \cdot 53^1 \cdot 5^1$. These coefficients respect conditions a), c) and d) from theorem 1 in [2], resulting in a valid 5-PP.

The values of *l*, leading to valid values of $Q$, are factorized as:

$$l = 2^{\alpha_{l,1}} \cdot 53^{\alpha_{l,2}} \cdot 5^{\alpha_{l,3}}, \qquad (175)$$

so that the factorization of $Q$ is:

$$Q = 2^{\alpha_{l,1}+3} \cdot 53^{\alpha_{l,2}} \cdot 5^{\alpha_{l,3}-2}. \qquad (176)$$

Conditions (166), (167a) and (168) have to be imposed.

Condition (166) can be written

$$2-\min\left\{\begin{array}{l}5,\\ \dfrac{5-4}{2}+\\ +\dfrac{\min\left\{1+4,\alpha_{l,1}+5+3+2,2\alpha_{l,1}+10+2,3\alpha_{l,1}+15\right\}}{2},\\ 1+\min\left\{2+1,\alpha_{l,1}+5\right\},\\ -2+\min\left\{3+2,\alpha_{l,1}+5+2+1,2\alpha_{l,1}+10+1\right\},\\ -4+\min\left\{1+4+1,\alpha_{l,1}+5+3+2,2\alpha_{l,1}+10+2+2,3\alpha_{l,1}+15\right\}\end{array}\right\}\leq \alpha_{l,1}\leq 2$$

$$\Leftrightarrow 2-\min\left\{\begin{array}{l}5,\\ \dfrac{1+\min\left\{5,\alpha_{l,1}+10,2\alpha_{l,1}+12,3\alpha_{l,1}+15\right\}}{2},\\ 1+\min\left\{3,\alpha_{l,1}+5\right\},\\ -2+\min\left\{5,\alpha_{l,1}+8,2\alpha_{l,1}+11\right\},\\ -4+\min\left\{6,\alpha_{l,1}+10,2\alpha_{l,1}+14,3\alpha_{l,1}+15\right\}\end{array}\right\}\le \alpha_{l,1}\le 2$$

$$\Leftrightarrow 2-\min\left\{5,3,4,3,2\right\}\le \alpha_{l,1}\le 2 \Leftrightarrow 0\le \alpha_{l,1}\le 2 \qquad (177)$$

Condition (167a) can be written as

$$1-\min\left\{\begin{array}{l}1,\\ \dfrac{1-2}{2}+\\ +\dfrac{\min\left\{1+2,\alpha_{l,2}+1+1+1,2\alpha_{l,2}+2+1,3\alpha_{l,2}+3\right\}}{2},\\ \min\left\{1,\alpha_{l,2}+1\right\},\\ -1+\min\left\{1+1,\alpha_{l,2}+1+1,2\alpha_{l,2}+2\right\},\\ -2+\min\left\{1+2,\alpha_{l,2}+1+1+1,2\alpha_{l,2}+2+1,3\alpha_{l,2}+3\right\}\end{array}\right\}\le \alpha_{l,2}\le 1$$

$$\Leftrightarrow 1-\min\left\{\begin{array}{l}1,\\ \dfrac{-1+\min\left\{3,\alpha_{l,2}+3,2\alpha_{l,2}+3,3\alpha_{l,2}+3\right\}}{2},\\ \min\left\{1,\alpha_{l,2}+1\right\},\\ -1+\min\left\{2,\alpha_{l,2}+2,2\alpha_{l,2}+2\right\},\\ -2+\min\left\{3,\alpha_{l,2}+3,2\alpha_{l,2}+3,3\alpha_{l,2}+3\right\}\end{array}\right\}\le \alpha_{l,2}\le 1$$

$$\Leftrightarrow 1-\min\left\{1,1,1,1,1\right\}\le \alpha_{l,2}\le 1 \Leftrightarrow 0\le \alpha_{l,2}\le 1 \qquad (178)$$

From (168), we have

$$\alpha_{l,3}=2 \qquad (179)$$

From (177) - (179), we see that there are 6 possible values for $l$ and, consequently, for $Q$, given in Table 13. The possible values for the vector $S$ for this interleaver, for which $Q_s=8$ (the minimum value of $Q$) are given in Table 14.

**Table 13**. Possible values for $Q$ in the 5-PP with $K=1696$, $f_1=55$, $f_2=954$, $f_3=1272$, $f_4=1484$ and $f_5=1060$.

| $l$ | 25 | 50 | 100 | 1325 | 2650 | 5300 |
|---|---|---|---|---|---|---|
| $Q$ | 8 | 16 | 32 | 424 | 848 | 1696 |

**Table 14**. Equivalent ARP interleavers with $P=f_1$, $Q_s=8$, $S(0)=S(4)=0$.

| $K$ | $f_1$ | $f_2$ | $f_3$ | $f_4$ | $f_5$ | $S(1)$ | $S(2)$ | $S(3)$ | $S(5)$ | $S(6)$ | $S(7)$ |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 1696 | 55 | 954 | 1272 | 1484 | 1060 | 1378 | 424 | 106 | 1378 | 424 | 106 |

Example 10:

Let $K = 216 = 2^3 \cdot 3^3$ and the 5-PP coefficients $f_1 = 11$, $f_2 = 36 = 2^2 \cdot 3^2$, $f_3 = 144 = 2^4 \cdot 3^2$, $f_4 = 162 = 2^1 \cdot 3^4$ și $f_5 = 54 = 2^1 \cdot 3^3$. These coefficients respect conditions a), c) and d) from theorem 1 in [2], resulting in a valid 5-PP.

The values of *l*, leading to valid values of $Q$, are factorized as:

$$l = 2^{\alpha_{l,1}} \cdot 3^{\alpha_{l,2}} \cdot 5^{\alpha_{l,3}}, \qquad (180)$$

so that the factorization of $Q$ is:

$$Q = 2^{\alpha_{l,1}+2} \cdot 3^{\alpha_{l,2}} \cdot 5^{\alpha_{l,3}-1}. (181)$$

Conditions (166), (167a) and (168) have to be inposed.

Condition (166) can be written

$$1 - \min\left\{\begin{array}{l} 3, \\ \frac{3-2}{2} + \\ + \frac{\min\left\{2+2, \alpha_{l,1}+3+4+1, 2\alpha_{l,1}+6+1, 3\alpha_{l,1}+9\right\}}{2}, \\ 1 + \min\left\{1+1, \alpha_{l,1}+3\right\}, \\ -1 + \min\left\{4+1, \alpha_{l,1}+3+1+1, 2\alpha_{l,1}+6+1\right\}, \\ -2 + \min\left\{2+2+1, \alpha_{l,1}+3+4+1, 2\alpha_{l,1}+6+1+2, 3\alpha_{l,1}+9\right\} \end{array}\right\} \le \alpha_{l,1} \le 1$$

$$1 - \min\left\{\begin{array}{l} 3, \\ \frac{1 + \min\left\{4, \alpha_{l,1}+8, 2\alpha_{l,1}+7, 3\alpha_{l,1}+9\right\}}{2}, \\ 1 + \min\left\{2, \alpha_{l,1}+3\right\}, \\ -1 + \min\left\{5, \alpha_{l,1}+5, 2\alpha_{l,1}+7\right\}, \\ -2 + \min\left\{5, \alpha_{l,1}+8, 2\alpha_{l,1}+9, 3\alpha_{l,1}+9\right\} \end{array}\right\} \le \alpha_{l,1} \le 1$$

$$\Leftrightarrow 1 - \min\left\{3, 2.5, 3, 4, 3\right\} \le \alpha_{l,1} \le 1 \Leftrightarrow 0 \le \alpha_{l,1} \le 1 \qquad (182)$$

Condition (167a) can be written as

$$3 - \min\left\{\begin{array}{l} 3, \\ \frac{3-6}{2} + \\ + \frac{\min\left\{2+6, \alpha_{l,2}+3+2+3, 2\alpha_{l,2}+6+4, 3\alpha_{l,2}+9\right\}}{2}, \\ \min\left\{4, \alpha_{l,2}+3\right\}, \\ -3 + \min\left\{2+3, \alpha_{l,2}+3+3+1, 2\alpha_{l,2}+6\right\}, \\ -6 + \min\left\{2+6, \alpha_{l,2}+3+2+3, 2\alpha_{l,2}+6+4, 3\alpha_{l,2}+9\right\} \end{array}\right\} \le \alpha_{l,2} \le 3$$

$$3-\min\left\{\begin{matrix}3,\\ \dfrac{-3+\min\{8,\alpha_{l,2}+8,2\alpha_{l,2}+10,3\alpha_{l,2}+9\}}{2},\\ \min\{4,\alpha_{l,2}+3\},\\ -3+\min\{5,\alpha_{l,2}+7,2\alpha_{l,2}+6\},\\ -6+\min\{8,\alpha_{l,2}+8,2\alpha_{l,2}+10,3\alpha_{l,2}+9\}\end{matrix}\right\}\le\alpha_{l,2}\le 3$$

$$\Leftrightarrow 3-\min\{3,2.5,3,2,2\}\le\alpha_{l,2}\le 3 \Leftrightarrow 1\le\alpha_{l,2}\le 3 \qquad (183)$$

From (168), we have

$$\alpha_{l,3}=1 \qquad (184)$$

From (182) - (184) we see that there are 6 possible values for $l$ and, consequently, for $Q$, given in Table 15. The possible values for the vector $S$ for this interleaver, for which $Q_s=12$ (the minimum value of $Q$) are given in Table 16.

**Table 15**. Possible values for $Q$ in the 5-PP with $K=216$, $f_1=11$, $f_2=36$, $f_3=144$, $f_4=162$ and $f_5=54$.

| $l$ | 15 | 30 | 45 | 90 | 135 | 270 |
|---|---|---|---|---|---|---|
| $Q$ | 12 | 24 | 36 | 72 | 108 | 216 |

**Table 16**. Equivalent ARP interleavers with $P=f_1$, $Q_s=12$, $S(0)=S(2)=S(3)=S(6)=S(8)=S(11)=0$.

| $K$ | $f_1$ | $f_2$ | $f_3$ | $f_4$ | | $S(1)$ | $S(4)$ | $S(5)$ | $S(7)$ | $S(9)$ | $S(10)$ |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 216 | 11 | 36 | 144 | 162 | 54 | 180 | 72 | 108 | 72 | 108 | 72 |

Example 11:

Let $K=4802=2^1\cdot 7^4$ and the 5-PP coefficients $f_1=10=2^1\cdot 5^1$, $f_2=70=2^1\cdot 5^1\cdot 7^1$, $f_3=7=7^1$, $f_4=14=2^1\cdot 7^1$ şi $f_5=196=2^2\cdot 7^2$. These coefficients respect conditions b) and c) from theorem 1 in [2], resulting in a valid 5-PP.

The values of $l$, leading to valid values of $Q$, are factorized as:

$$l=2^{\alpha_{l,1}}\cdot 7^{\alpha_{l,2}}\cdot 5^{\alpha_{l,3}}, \qquad (185)$$

so that the factorization of $Q$ is:

$$Q=2^{\alpha_{l,1}-1}\cdot 7^{\alpha_{l,2}+2}\cdot 5^{\alpha_{l,3}-1}. \qquad (186)$$

Conditions (166), (167a) and (168) have to be inposed.

Condition (166) can be written

$$2-\min\left\{\begin{array}{l}1,\\ \frac{1-4}{2}+\\ +\frac{\min\left\{1+4,\alpha_{l,1}+1+0+2,2\alpha_{l,1}+2+1,3\alpha_{l,1}+3\right\}}{2},\\ 1+\min\left\{1+1,\alpha_{l,1}+1\right\},\\ -2+\min\left\{0+2,\alpha_{l,1}+1+2+1,2\alpha_{l,1}+2+1\right\},\\ -4+\min\left\{1+4+1,\alpha_{l,1}+1+0+2,2\alpha_{l,1}+2+1+2,3\alpha_{l,1}+3\right\}\end{array}\right\}\le\alpha_{l,1}\le 2$$

$$2-\min\left\{\begin{array}{l}1,\\ \frac{-3+\min\left\{5,\alpha_{l,1}+3,2\alpha_{l,1}+3,3\alpha_{l,1}+3\right\}}{2},\\ 1+\min\left\{2,\alpha_{l,1}+1\right\},\\ -2+\min\left\{2,\alpha_{l,1}+4,2\alpha_{l,1}+3\right\},\\ -4+\min\left\{6,\alpha_{l,1}+3,2\alpha_{l,1}+5,3\alpha_{l,1}+3\right\}\end{array}\right\}\le\alpha_{l,1}\le 2$$

$$\Leftrightarrow 2-\min\left\{\begin{array}{l}1,\\ \frac{-3+\min\left\{5,\alpha_{l,1}+3,2\alpha_{l,1}+3,3\alpha_{l,1}+3\right\}}{2},\\ 1+\min\left\{2,\alpha_{l,1}+1\right\},\\ 0,\\ -4+\min\left\{6,\alpha_{l,1}+3,2\alpha_{l,1}+5,3\alpha_{l,1}+3\right\}\end{array}\right\}\le\alpha_{l,1}\le 2 \qquad (187)$$

Because

$$\min\left\{\begin{array}{l}1,\\ \frac{-3+\min\left\{5,\alpha_{l,1}+3,2\alpha_{l,1}+3,3\alpha_{l,1}+3\right\}}{2},\\ 1+\min\left\{2,\alpha_{l,1}+1\right\},\\ 0,\\ -4+\min\left\{6,\alpha_{l,1}+3,2\alpha_{l,1}+5,3\alpha_{l,1}+3\right\}\end{array}\right\}\le 0,$$

The double inequality in (187) is fulfilled only for

$\alpha_{l,1}=2$ (188)

Condition (167a) can be written

$$2-\min\left\{\begin{array}{l}4,\\ \frac{4-4}{2}+\\ +\frac{\min\left\{1+4,\alpha_{l,2}+4+1+2,2\alpha_{l,2}+8+1,3\alpha_{l,2}+12\right\}}{2},\\ \min\left\{1,\alpha_{l,2}+4\right\},\\ -2+\min\left\{1+2,\alpha_{l,2}+4+2,2\alpha_{l,2}+8\right\},\\ -4+\min\left\{1+4,\alpha_{l,2}+4+1+2,2\alpha_{l,2}+8+1,3\alpha_{l,2}+12\right\}\end{array}\right\}\le\alpha_{l,2}\le 2$$

$$2-\min\left\{\begin{array}{l}4,\\ \frac{\min\left\{5,\alpha_{l,2}+7,2\alpha_{l,2}+9,3\alpha_{l,2}+12\right\}}{2},\\ \min\left\{1,\alpha_{l,2}+4\right\},\\ -2+\min\left\{3,\alpha_{l,2}+6,2\alpha_{l,2}+8\right\},\\ -4+\min\left\{5,\alpha_{l,2}+7,2\alpha_{l,2}+9,3\alpha_{l,2}+12\right\}\end{array}\right\}\le\alpha_{l,2}\le 2$$

$$\Leftrightarrow 2-\min\left\{4,\frac{5}{2},1,1,1\right\}\le\alpha_{l,2}\le 2 \Leftrightarrow 1\le\alpha_{l,2}\le 2 \qquad (189)$$

From (168), we have

$$\alpha_{l,3}=1 \qquad (190)$$

From (1882) - (190) we see that there are 2 possible values for $l$ and, consequently, for $Q$, given in Table 17. As the minimum value of $Q$ is too large, namely $Q_s=686$, we have not given the possible values for the vector $S$ for this interleaver.

**Table 17**. Possible values for $Q$ in the 4-PP with $K=4802$, $f_1=10$, $f_2=70$, $f_3=7$, $f_4=14$ and $f_5=196$.

| $l$ | 140 | 980 |
|---|---|---|
| $Q$ | 686 | 4802 |

<u>Example 12:</u>

Let $K=1225=5^2\cdot 7^2$ and the 5-PP coefficients $f_1=12=2^2\cdot 3^1$, $f_2=525=3^1\cdot 5^2\cdot 7^1$, $f_3=140=2^2\cdot 5^1\cdot 7^1$, $f_4=245=5^1\cdot 7^2$ și $f_5=245=5^1\cdot 7^2$. These coefficients respect conditions a) and c) from theorem 1 in [2], resulting in a valid 5-PP.

The values of $l$, leading to valid values of $Q$, are factorized as:

$$l=2^{\alpha_{l,1}}\cdot 5^{\alpha_{l,2}}\cdot 7^{\alpha_{l,3}}, \qquad (191)$$

so that the factorization of $Q$ is:

$$Q=2^{\alpha_{l,1}}\cdot 5^{\alpha_{l,2}}\cdot 7^{\alpha_{l,3}}=l \qquad (192)$$

In the following, we consider that $\alpha_{K,1}=0$, $\alpha_{f_2,1}=0$, $\alpha_{f_4,1}=0$ and $\alpha_{f_5,1}=0$.

Conditions (170), (171) and (172a) have to be imposed.

From (170), we have

$\alpha_{l,1} = 0$. (193)

Condition (171) can be written

$$1+1-\min\left\{\begin{array}{l} 2, \\ \frac{2-2-3}{2}+ \\ +\frac{\min\left\{2+2+3,\alpha_{l,2}+2+1+1+2,2\alpha_{l,2}+4+1+1,3\alpha_{l,2}+6\right\}}{2}, \\ \min\left\{1,\alpha_{l,2}+2\right\}, \\ -1-1+\min\left\{1+1+1,\alpha_{l,2}+2+1+1,2\alpha_{l,2}+4\right\}, \\ -2-2+ \\ +\min\left\{2+2+2,\alpha_{l,2}+2+1+1+1,2\alpha_{l,2}+4+1,3\alpha_{l,2}+6\right\} \end{array}\right\} \le \alpha_{l,2} \le 1+1$$

$$2-\min\left\{\begin{array}{l} 2, \\ \frac{-3+\min\left\{7,\alpha_{l,2}+6,2\alpha_{l,2}+6,3\alpha_{l,2}+6\right\}}{2}, \\ \min\left\{1,\alpha_{l,2}+2\right\}, \\ -2+\min\left\{3,\alpha_{l,2}+4,2\alpha_{l,2}+4\right\}, \\ -4+\min\left\{6,\alpha_{l,2}+5,2\alpha_{l,2}+5,3\alpha_{l,2}+6\right\} \end{array}\right\} \le \alpha_{l,2} \le 2$$

$$\Leftrightarrow 2-\min\left\{2,\frac{3}{2},1,1,1\right\} \le \alpha_{l,2} \le 2 \Leftrightarrow 1 \le \alpha_{l,2} \le 2 \qquad (194)$$

Condition (172a) can be written

$$2-\min\left\{\begin{array}{l} 2, \\ \frac{2-4}{2}+\frac{\min\left\{1+4,\alpha_{l,3}+2+1+2,2\alpha_{l,3}+4+2,3\alpha_{l,3}+6\right\}}{2}, \\ \min\left\{2,\alpha_{l,3}+2\right\}, \\ -2+\min\left\{1+2,\alpha_{l,3}+2+2,2\alpha_{l,3}+4\right\}, \\ -4+\min\left\{1+4,\alpha_{l,3}+2+1+2,2\alpha_{l,3}+4+2,3\alpha_{l,3}+6\right\} \end{array}\right\} \le \alpha_{l,3} \le 2$$

$$2-\min\left\{\begin{array}{l} 2, \\ -1+\frac{\min\left\{5,\alpha_{l,3}+5,2\alpha_{l,3}+6,3\alpha_{l,3}+6\right\}}{2}, \\ \min\left\{2,\alpha_{l,3}+2\right\}, \\ -2+\min\left\{3,\alpha_{l,3}+4,2\alpha_{l,3}+4\right\}, \\ -4+\min\left\{5,\alpha_{l,3}+5,2\alpha_{l,3}+6,3\alpha_{l,3}+6\right\} \end{array}\right\} \le \alpha_{l,3} \le 2$$

$$\Leftrightarrow 2-\min\left\{2,2,2,1,1\right\} \le \alpha_{l,3} \le 2 \Leftrightarrow 1 \le \alpha_{l,3} \le 2 \qquad (195)$$

From (193) - (195), we see that there are 4 possible values for $l$ and, consequently, for $Q$, given in Table 18. As the minimum value of $Q$ is too large, namely $Q_s = 35$, we have not given the possible values for the vector $S$ for this interleaver.

**Table 18**. Possible values for $Q$ in the CPP with $K = 1225$, $f_1 = 12$, $f_2 = 525$, $f_3 = 140$, $f_4 = 245$ and $f_5 = 245$.

| $l$ | 35 | 175 | 245 | 1225 |
|---|---|---|---|---|
| $Q$ | 35 | 175 | 245 | 1225 |

# 5. Conclusion and final remarks

In this paper, the conditions in [1] for a QPP interleaver to be expressed as an ARP interleaver were extended to PP interleavers of any degree. These conditions were customized for PP interleavers of degree 3, 4 or 5, whose coefficients satisfy the sufficient conditions from [2]. It was shown that these PP interleavers can always be expressed as special cases of an ARP interleaver in which the values of the periodic shifts follow the nonlinear term of the PP interleaver function. Therefore, the ARP interleaver is a sufficient permutation model to design TCs with minimum Hamming distances and, therefore, with the achievable asymptotic performance of any of the above mentioned PP interleavers.

We remark that PLPP based interleavers composed of $L$ LPPs were introduced in [23]. It was shown that an ARP is actually a particular PLPP with the linear terms coefficients of the component LPPs equal to the value of *P* from (1) and $L = Q$. Sufficient conditions for a *m*-PP to be written as a PLPP were also given. These conditions are more restrictive for *m*-PP than those in [2], meaning that the coefficients $f_j$, with $j = 2,...,m$, must have 2 as factor, when *K* is even, condition that is not required by the condition c) of Theorem 1 of [2]. In general, the minimum number of LPPs of the PLPP equivalent to a *m*-PP, $L$, is half of the minimum value of $Q$ for the ARP equivalent to that *m*-PP, when the length of the interleaver is even. Thus, the number of values to be stored is $2 \cdot L$ for the equivalent PLPP and $Q + 1 = 2 \cdot L + 1$ for the equivalent ARP. Considering values equal to 0 for some ARP parameters leads to approximately the same storage requirements, and the same computational complexity (i.e., a multiplication, an addition and a modulo *K* operation for each of the *K* values of the permutation corresponding to the interleaver).

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